Week 2 (Fourier series)

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* Fourier series: Is a mathematical method used To analysis periodic function (signals) into sine And cosine waves with different frequencies.

* General equation:

$$F(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n w_0 t) + b_n \sin(n w_0 t) \right)$$

a_0, a_n, b_n: Is called Fourier coefficients And can be calculated as following:

[1] $$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} F(t)\, dt$$

[2] $$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(n w_0 t)\, dt$$

[3] $$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(n w_0 t)\, dt$$

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w_0: Angular velocity (frequency) → rad/s $= \frac{2\pi}{T}$

T: Time taken to make one cycle → s $= T_2 - T_1$

مهم

* لو الدالة زوجية (even)

$b_n = 0$

وحسب a_0, a_n

لو فردية (odd)

$a_0 = a_n = 0$

وحسب b_n

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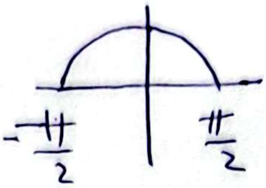
Ex 1

$f(t) = \cos(t)$



where $-\frac{\pi}{2} < x < \frac{\pi}{2}$

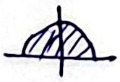
Find F.S (Fourier Series)

Sol

$\because f(t) = \cos(t) \rightarrow$ even function $\rightarrow \therefore \boxed{b_n = 0}$

why ??

$f(-t) = f(t)$



$T = \frac{\pi}{2} - (-\frac{\pi}{2}) = \boxed{\pi}$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = \boxed{2}$ rad/s

1 $a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{T/2} f(t)\,dt$

$a_0 = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\pi/2} \cos(t)\,dt$

$= \frac{1}{\pi} * \sin(t)\Big|_{-\pi/2}^{\pi/2} = \frac{1}{\pi}\left(\sin(\frac{\pi}{2}) - \sin(-\frac{\pi}{2})\right)$

$= \frac{1}{\pi}(1-(-1)) = \boxed{\frac{2}{\pi}}$

2 $a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)\,dt$

$= \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\pi/2} \cos(t)\cos(2nt)\,dt \rightarrow$ $\cos(At)\cos(Bt) = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$

$= \frac{\cancel{2}}{\pi} * \frac{1}{\cancel{2}}\left(\int_{-\frac{\pi}{2}}^{\pi/2} \cos(2n+1)t\,dt + \int_{-\frac{\pi}{2}}^{\pi/2} \cos(2n-1)\,dt\right.$

$= \frac{1}{\pi} * \left[\frac{\sin(2n+1)t}{2n+1}\Big|_{-\frac{\pi}{2}}^{\pi/2} + \frac{\sin(2n-1)t}{2n-1}\Big|_{-\frac{\pi}{2}}^{\pi/2}\right]$

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$$\frac{1}{\pi} * \left[\frac{1}{2n+1} * \left(\sin\left(\frac{(2n+1)\pi}{2}\right) - \sin\left(\frac{(2n+1) * -\pi}{2}\right) \right) \right.$$

$$\left. + \frac{1}{2n-1} * \left(\sin\left(\frac{(2n-1)\pi}{2}\right) - \sin\left(\frac{(2n-1) * -\pi}{2}\right) \right) \right]$$

$$= \frac{1}{\pi} * \left[\frac{1}{2n+1} \left((-1)^n - (-1)^{n+1} \right) \right.$$

$$\left. + \frac{1}{2n-1} * \left((-1)^{n+1} - (-1)^n \right) \right]$$

Note

$\sin\left(\frac{(2n+1)\pi}{2}\right) = (-1)^n$

$\sin\left(\frac{(2n+1)-\pi}{2}\right) = (-1)^{n+1}$

$\sin\left(\frac{(2n-1)\pi}{2}\right) = (-1)^{n+1}$

$\sin\left(\frac{(2n-1)-\pi}{2}\right) = (-1)^n$

For $n = 1, 2$

$$a_1 = \frac{1}{\pi} \left[\frac{1}{2(1)+1} \left((-1)^1 - (-1)^{(1+1)} \right) \right.$$

$$\left. + \frac{1}{2(1)-1} * \left((-1)^{(1)+1} - (-1)^1 \right) \right]$$

$$= \frac{1}{\pi} \left[\frac{1}{3} (-1-1) + \frac{1}{1} (1+1) \right]$$

$$= \frac{1}{\pi} \left[\frac{-2}{3} + 2 \right] = \frac{1}{\pi} * \frac{4}{3} = \boxed{\frac{4}{3\pi}}$$

$$a_2 = \frac{1}{\pi} \left[\frac{1}{2(2)+1} \left((-1)^2 - (-1)^{(2+1)} \right) \right.$$

$$\left. + \frac{1}{2(2)-1} * \left((-1)^{(2)+1} - (-1)^{(2)} \right) \right] = \frac{1}{\pi} \left[\frac{1}{5} * (1+1) \right.$$

$$\left. + \frac{1}{3} (-1-1) \right] = \frac{1}{\pi} \left(\frac{2}{5} - \frac{2}{3} \right)$$

$$= \frac{1}{\pi} * -\frac{4}{15} = \boxed{-\frac{4}{15\pi}}$$

$$\therefore f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n w_0 t)$$

n=1, wo=2   n=2, wo=2

$$= \frac{2}{\pi} + \frac{4}{3\pi} \cos(2t) - \frac{4}{15\pi} \cos(4t) \;---\; \text{and so on}$$

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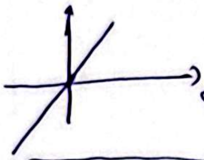
Ex2 $\quad g(t) = t \qquad -\pi < x < \pi$

Find F.S fourier Series

Sol

$g(t) = t \rightarrow$  $\rightarrow$ odd function

$\boxed{g(-t) = -g(t) = -t} \rightarrow$ حلو

$\because$ Function is odd $\rightarrow \therefore \boxed{a_0 = a_n = 0}$

$T = \pi - (-\pi) = \pi + \pi = \boxed{2\pi}$

$\omega_0 = \dfrac{2\pi}{2\pi} = \boxed{1 \text{ rad/s}}$

$\boxed{1}$ $b_n = \dfrac{2}{T} \displaystyle\int_{-T/2}^{T/2} g(t) \sin(n\omega_0 t)\, dt$

$= \dfrac{2}{2\pi} \displaystyle\int_{-\frac{2\pi}{2}}^{2\pi/2} t \sin(n(1)t)\, dt$

$= \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} t \sin(nt)\, dt$

$= \dfrac{1}{\pi} \left[-\dfrac{t\cos(nt)}{n} \Big|_{-\pi}^{\pi} + \dfrac{\sin(nt)}{n^2} \Big|_{-\pi}^{\pi} \right.$

$= \dfrac{1}{\pi} \left[\left(-\dfrac{1}{n} \left(\pi \cos(n\pi) + \pi \cos(n \cdot -\pi) \right) \right) \right.$

$\left. + \dfrac{1}{n^2} \left(\sin(n\pi) - \sin(-n\pi) \right) \right]^{0}$ $\quad \boxed{\sin(180) = 0}$

$= \dfrac{1}{\pi} \left[-\dfrac{2\pi}{n} (-1)^n \right] = -\dfrac{2\pi}{\pi n} (-1)^n$

$= \boxed{-\dfrac{2}{n} (-1)^n}$

Note

تفاضل	تكامل
t	$\sin(nt)$
1	$-\dfrac{\cos(nt)}{n}$
0	$-\dfrac{\sin(nt)}{n^2}$

$\oplus$ $\ominus$

$\cos(n\pi) = (-1)^n$

$\cos(-n\pi) = (-1)^n$

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$$\because \; b_n = -\frac{2}{n}(-1)^n$$

For $n = 1, 2, 3$

$$b_1 = -\frac{2}{(1)}(-1)^1 = \boxed{2}$$

$$b_2 = -\frac{2}{(2)}(-1)^2 = \boxed{-1}$$

$$b_3 = -\frac{2}{(3)}(-1)^3 = \boxed{\frac{2}{3}}$$

$$g(t) = \sum b_n \sin(n\omega_0 t)$$

$$= 2\sin(t) \cancel{+} \sin(2t) + \frac{2}{3}\sin(3t) \ldots \text{az som}$$

Ex3 $f(x) = \pi^2 - x^2 \qquad -\pi < x < \pi$

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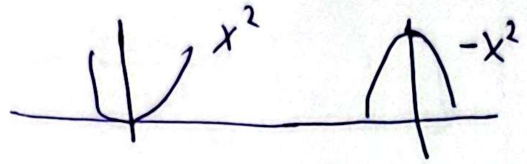
Find F.S (fourier series)

sol

$$F(x) = \pi^2 - x^2 \rightarrow F(-x) = \pi^2 - (-x)^2 = \pi^2 - x^2 = f(x)$$

دورقم ثابت $\leftarrow$

The function is even $\boxed{\therefore b_n = 0}$



$$T = \pi - (-\pi) = \boxed{2\pi}$$

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = \boxed{1 \text{ rad/s}}$$

$\boxed{1}$ $$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\,dx = \frac{1}{2\pi}\int_{-\frac{2\pi}{2}}^{\frac{2\pi}{2}} (\pi^2 - x^2)\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\pi^2 - x^2)\,dx = \left(\pi^2 x\Big|_{-\pi}^{\pi} - \frac{x^3}{3}\Big|_{-\pi}^{\pi}\right) * \frac{1}{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(\pi^2(\pi - (-\pi))\right) - \left(\frac{1}{3}(\pi^3 - (-\pi)^3)\right)\right]$$

$$= \frac{1}{2\pi}\left[2\pi^3 - \frac{1}{3} * 2\pi^3\right] = \frac{1}{2\pi}\left[\frac{4}{3}\pi^3\right]$$

$$= \boxed{\frac{2\pi^2}{3}}$$

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$\boxed{2}$ $a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos(n w_0 x)\,dx$

$= \frac{2}{2\pi}\int_{-2\pi/2}^{2\pi/2} (\pi^2 - x^2)\cos(n(1)x)\,dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \pi^2 \cos(nx)\,dx - \int_{-\pi}^{\pi} x^2\cos(nx)\,dx\right]$

$= \frac{1}{\pi}\left[\cancel{\pi^2 \frac{\sin(nx)}{n}\Big|_{-\pi}^{\pi}}^{\,0} - \left(\cancel{\frac{x^2\sin(nx)}{n}\Big|_{-\pi}^{\pi}}^{\,0} + \frac{2x\cos(nx)}{n^2}\Big|_{-\pi}^{\pi} - \cancel{\frac{2\sin(nx)}{n^3}\Big|_{-\pi}^{\pi}}_{\,0}\right)\right]$

لأن (أدرفي في $Sin(180$ = صفر

$= \frac{1}{\pi}\left[0 - \left(0 + \left(\frac{2\pi(\cos(n\pi))}{n^2} - \frac{2(-\pi)\cos(-n\pi)}{n^2} - 0\right)\right]\right.$

$= \frac{1}{\cancel{\pi}}\left[-\frac{4\cancel{\pi}}{n^2}(-1)^n\right]$

$= -\frac{4}{n^2}(-1)^n$

For $n = 1, 2, 3$

$a_1 = -\frac{4}{(1)^2}(-1)^{(1)} = \boxed{4}$

$a_2 = -\frac{4}{(2)^2}(-1)^{(2)} = \boxed{-1}$

$a_3 = -\frac{4}{(3)^2}(-1)^3 = \boxed{\frac{4}{9}}$

note

تفاضل	تكامل
x^2	$\cos nx$
$2x$	$\frac{\sin(nx)}{n}$
2	$-\frac{\cos(nx)}{n^2}$
0	$-\frac{\sin(nx)}{n^3}$

(+) (−) (+)

ضد بالالك

$\frac{2\pi\cos(n\pi)}{n^2} + \frac{2\pi\cos(-n\pi)}{n^2}$

$= \frac{4\pi\cos(n\pi)}{n^2}$

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n w_0 x)$

$= \frac{2\pi^2}{3} + 4\cos(x) - \cos(2x) + \frac{4}{9}\cos(3x) - - - - -$