Fundamentals of engineering

NCEES FE ELECTRICAL EXAM

Your Path to Success

By Eng/Abdullah Ibrahim

مقدمة وأنواع اختبار مبادئ الهندسة

مقدمة وأنواع اختبار مبادئ الهندسة

يعد اختبار أساسيات الهندسة اختباراً شاملاً للأساسيات الهندسية، ويهدف الى قياس الكفاءة والمعرفة الهندسية الأساسية للمهندسين الراغبين في الحصول على درجة مهندس مشارك، كما يمكن دخول الاختبار والتجهيز لاجتيازه لمن هم على وشك التخرج، ولحديثي التخرج، وكذلك المهندسين على رأس العمل، ويقدم الاختبار باللغة الإنجليزية وبشكل محوسب.

الاختبار في التخصص الهندسي

الذي يغطي المجالات الأساسية لكل تخصص من التخصصات الهندسية التالية:

1 الهندسة الكهربائية:

ويغطي هذا الجزء المجالات الهندسية في التخصص، حيث تبلغ عدد أسئلة الاختبار 50 سؤالا ويستغرق ثلاث ساعات، ويغطي هذا الاختبار المواضيع التالية:

- الدوائر الكهربائية
- أنظمة الطاقة
- الكهرومغناطيسية
- أنظمة التحكم
- الاتصالات
- معالجة الإشارات
- الإلكترونيات
- الأنظمة الرقمية
- أنظمة الكمبيوتر

وتعتمد الهيئة الاختبارات التالية:

الأول: اختبار الهيئة السعودية للمهندسين (اختبار أساسيات الهندسة) SCE - FE

ويحتوي الاختبار على جزئين:

الاختبار الهندسي العام

الذي يغطي المجالات الهندسية العامة التي تشترك فيها جميع التخصصات الهندسية، حيث تبلغ عدد أسئلة الاختبار 90 سؤالا ويستغرق ثلاث ساعات ويغطي هذا الاختبار المجالات التالية:

- الرياضيات.
- الاحصاء والاحتمالات.
- مبادئ الكمبيوتر.
- الستاتيكا والديناميكا.
- الكيمياء.
- الديناميكا الحرارية.
- ميكانيكا الموائع.
- علوم وهندسة المواد.
- الكهرباء والمغناطيسية.
- الرسم الهندسي.
- الاقتصاد الهندسي.

- ادارة المشاريع.
- اخلاقيات المهنة.

المهارات العامة:

- مهارات التفكير التحليلى والتفكير النقدي.
- مواكبة المستجدات في مجال التخصص وخارجه.
- مهارات التواصل.
- مهارات العمل ضمن فريق متعدد التخصصات.
- مهارات ادارة الوقت.
- مهارات التطوير المستمر.
- أدراك مفهوم الاستدامة والتأثير البيئي والقانون والاجتماع للحلول والتصاميم الهندسية.

مقدمة وأنواع اختبار مبادئ الهندسة

أساسيـات الهـنـدسـة Fundamentals of Engineering (FE)

عدد مرات عقد اختبار أساسيات الهندسة:

يعقد الاختبار ثلاث مرات في العام، ويفتح التسجيل قبل موعد الاختبار بـ 30 يوماً تقريباً.

طبيعة اختبار أساسيات الهندسة:

يقدم اختبار الهيئة السعودية للمهندسين عن طريق منصات المركز الوطني للقياس باللغة الإنجليزية، وبشكل محوسب، ويحتوي على أسئلة من نوع متعددة الخيارات.

رسوم اختبار أساسيات الهندسة:

٨٠٠ ريال تدفع مباشرة للهيئة السعودية للمهندسين بالإضافة لرسوم أخرى تدفع للمركز الوطني للقياس

طريقة التسجيل لاختبار أساسيات الهندسة:

- التقدم بطلب حضور الاختبار عبر الصفحة الخاصة بالمهندس في الهيئة السعودية للمهندسين.
- سوف تصدر فاتورة خاصة بالهيئة بقيمة (800 ريال) يتم دفع قيمة الفاتورة عن طريق خدمة سداد الخاصة بالهيئة السعودية للمهندسين.
- بعد استلام الطلب سوف تقوم الهيئة السعودية للمهندسين بالتحقق من الطلب، وبعد الموافقة يمكن للمهندس التسجيل في موقع قياس التابع لهيئة تقويم التعليم والتدريب ودفع الرسوم الخاص بهم.

مقدمة وأنواع اختبار مبادئ الهندسة

الاختبار الأمريكي المقدم من المجلس الوطني الأمريكي لمختبري الهندسة والمساحة (NCEES)

حيث يقدم عن طريق (NCEES) الاختبارات التالية:

1- الهندسة الكهربائية وهندسة الحاسب
2- الهندسة الميكانيكية
3- الهندسة المدنية
4- الهندسية الكيميائية
5- الهندسة الصناعية وهندسة النظم
6- الهندسة البيئية
7- أخرى (لجميع التخصصات الهندسية الأخرى)

متطلبات الدخول التسجيل لدخول اختبار (NCEES) (FE,FS)

- الحصول على عضوية الهيئة السعودية للمهندسين وتكون سارية (كمهندس هوية سارية المفعول.
- دفع رسوم الاختبار.
- اختيار التخصص الذي يرغب التسجيل فيه ويجب أن يكون مطابقاً لتخصصه الدراسي.

الخطوات المتبعة من أجل التأهيل لحضور اختبار الأساسيات المقدم من (NCEES)

- التقدم بطلب حضور الاختبار عبر الصفحة الخاصة بالمهندس في الهيئة السعودية للمهندسين
- سوف تصدر فاتورة خاصة بالهيئة بقيمة (١٥٠٠ ريال) يتم دفع قيمة الفاتورة عن طريق خدمة سداد الخاصة بالهيئة السعودية للمهندسين
- التسجيل في موقع (NCEES) https://ncees.org/supplemental/launch-login/
- دفع فاتورة (NCEES) وقيمتها (٢٠٠$) دولار أمريكي
- ارسال ايميل لـ nceesexams@saudieng.sa في حال تم دفع الفاتورة وإضافة الرقم المرجعي الخاص بـ NCEES (xx-xxx-xx)
- بعد استلام الطلب سوف تقوم الهيئة السعودية للمهندسين بالتحقق من الطلب، وبعد الموافقة يمكن للمهندس الدخول على صفحته الخاصة بموقع (NCEES) وتحديد موعد ومكان الاختبار

مقدمة وأنواع اختبار مبادئ الهندسة

الاختبار الأمريكي المقدم من المجلس الوطني الأمريكي لمختبري الهندسة والمساحة (NCEES)

حيث يقدم عن طريق (NCEES) الاختبارات التالية:

1- الهندسة الكهربائية وهندسة الحاسب
2- الهندسة الميكانيكية
3- الهندسة المدنية
4- الهندسية الكيميائية
5- الهندسة الصناعية وهندسة النظم
6- الهندسة البيئية
7- أخرى (لجميع التخصصات الهندسية الأخرى)

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العضوية والترقيات والدرجات المهنية

العضوية والترقيات والدرجات المهنية

أنواع العضويات

أنواع العضويات

اعتماد مهني (مهندس- تقني الهندسة - معماري - تقني عمارة)

تفاصيل العضوية

عضوية التخصصات المساندة للهندسة (اخصائي)

تفاصيل العضوية

عضوية فني

تفاصيل العضوية

عضوية طالب

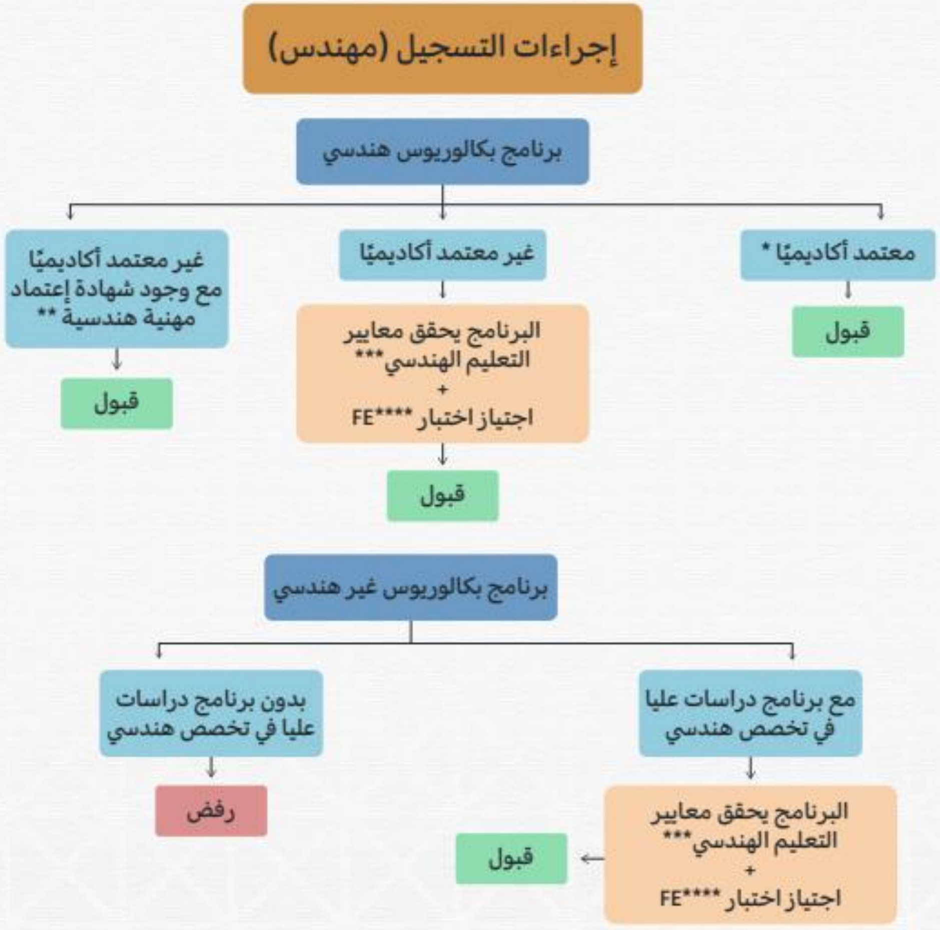
تفاصيل العضوية

عضوية مهتم بعلوم الهندسة

تفاصيل العضوية

العضوية والترقيات والدرجات المهنية

شكل رقم (1): إجراءات دراسة الطلبات لفئة مهندس



* برنامج مدته على الأقل ٤ سنوات معتمد من مؤسسة أو مجلس اعتماد أكاديمي محليًا أو عالميًا ويحقق معايير التعليم الهندسي.

** هيئة اعتماد مهنية هندسية أو نقابة هندسية أو مجلس هندسي في بلد التخرج ومعتمدة لدى الهيئة.

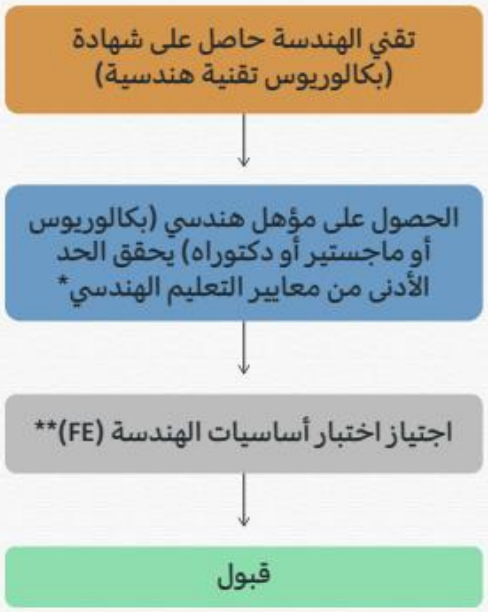
*** تدخل في دراسة الطلب جميع المقررات وجميع الدرجات التي تحصل عليها المتقدم على أن تشمل (العلوم الأساسية + مقررات هندسية في التحليل والتصميم الهندسي + المهارات التخصصية والتقنية في المجال الهندسي) وفقاً لمعايير التعليم الهندسي الواردة في الجدول (1).

**** اختبارFE: اختبار أساسيات الهندسة – Fundamentals of Engineering Exam

إجراءات ترقية فئة تقني الهندسة إلى فئة مهندس:

يتم دراسة ومراجعة كافة المستندات المطلوبة والتأكد من الاعتماد الأكاديمي الهندسي لكل طلب حسب الإجراءات الموضحة في الشكل رقم (3) والتأكد من استحقاق المتقدم لقب مهندس.

شكل رقم (3): إجراءات ترقية فئة تقني الهندسة إلى فئة مهندس



* برنامج لا تقل مدته عن 4 سنوات ومعتمد من مؤسسة أو مجلس اعتماد أكاديمي محليًا أو عالميًا ويحقق معايير التعليم الهندسي وفقًا للمعايير في الجدول رقم (1).

** اختبار أساسيات الهندسة – (FE) Fundamentals of Engineering Exam

أولاً: الدرجات المهنية:

تصنف الدرجات المهنية وفق الاتي:

(درجـــة مهندس/معماري - درجة مشـــارك - درجة محترف - درجة مستشـــار)

1- درجة مهندس/معماري

- **المتطلبات:**

1- الحصول على درجة بكالوريوس من برنامج أكاديمي معتمد في أحد التخصصات الهندسية/المعمارية من كلية أو برنامج محقق لمعايير التسجيل في الهيئة.

- **المهام والمسؤوليات:**

1- تقديم المساعدات الفنية لفريق العمل الذي يعمل معه.
2- التركيز على تطوير قدراته المهنية والفنية من خلال برامج التطوير المهني.

العضوية والترقيات والدرجات المهنية

2- درجة مشارك

المتطلبات:

1- الحصول على درجة البكالوريوس من برنامج أكاديمي معتمد في أحد التخصصات الهندسية/المعمارية من كلية أو برنامج محقق لمعايير التسجيل في الهيئة.

2- عضوية سارية المفعول لدى الهيئة.

3- اجتياز اختبار الأساسيات الهندسة/المعمارية (FE/FA).

4- سداد رسوم التصنيف.

المهام والمسؤوليات:

1- مراجعة التقارير والتصاميم تحت إشراف مهندس/معماري معتمد مهنيًا لا تقل درجته المهنية عن درجة محترف.

2- تقديم خدمات هندسية/معمارية (كالدراسات والتصاميم) في مجال تخصصه. تحت إشراف ومتابعة واعتماد معتمد مهنيًا ومصنّف لا تقل درجته المهنية عن درجة محترف.

3- ممارسة كافة المهام الهندسية/المعمارية تحت إشراف ومتابعة واعتماد من معتمد مهنيًا لا تقل درجته المهنية عن محترف.

4- التوقيع على التقارير والتصاميم بجانب معتمد مهنيًا لا تقل درجته المهنية عن محترف.

5- التركيز على تطوير قدراته المهنية والفنية من خلال برامج التطوير المهني.

العضوية والترقيات والدرجات المهنية

3- درجة محترف

المتطلبات:

1- الحصول على درجة البكالوريوس من برنامج أكاديمي معتمد في أحد التخصصات الهندسية/المعمارية من كلية أو برنامج محقق لمعايير التسجيل في الهيئة.
2- عضوية سارية المفعول لدى الهيئة.
3- خبرة موثقة في مجال تخصصه لا تقل عن خمس سنوات.
4- اجتياز اختبار الممارسة الهندسية/المعمارية (PE/PA).
5- سداد رسوم التصنيف.

المهام والمسؤوليات:

1- تقديم خدمات هندسية أو معمارية في مجال تخصصه الدقيق.
2- تقديم استشارات فنية في مجال تخصصه الدقيق.
3- اعتماد وختم والتوقيع على الوثائق مثل المخططات والتقارير والتصاميم في مجال تخصصه الدقيق.
4- تدريب المعتمدين مهنيًا والمصنفين على درجة مشارك فأقل.
5- عضوية اللجان الفنية في الهيئة.
6- التحكيم وتسوية المنازعات بعد الحصول على التأهيل المطلوب.
7- إدارة عقود الخدمات في مجال تخصصه.
8- تولي مهام و مسؤوليات قيادية.
9- المساهمة في المجال المعرفي في المجالات الهندسية/المعمارية.

4- درجة مستشار

المتطلبات:

1- الحصول على درجة البكالوريوس من برنامج أكاديمي معتمد في أحد التخصصات الهندسية/المعمارية من كلية أو برنامج محقق لمعايير التسجيل في الهيئة.

2- عضوية سارية المفعول لدى الهيئة.

3- خبرة ممارسة مهنية على درجة محترف لا تقل عن خمس سنوات، أو خبرة ممارسة مهنية لا تقل عن عشر سنوات بالإضافة لاجتياز اختبار الممارسة الهندسية/المعمارية (PE/PA) لمن لم يسبق له الاختبار.

4- سداد رسوم التصنيف.

المهام والمسؤوليات:

1- جميع المهام والصلاحيات المنصوص عليها لدرجة محترف.

2- التدريب والإشراف والتوجيه للمصنفين على درجات أقل لتطوير مسيرتهم المهنية.

3- المشاركة في اقتراح الخطط العامة لإعداد وتطوير القوى العاملة في المجالات الهندسية والمعمارية.

العضوية والترقيات والدرجات المهنية

كم أقل خبرة مطلوبة لاختبار PE؟

يجب على مقدم الطلب لاختبار PE أن تكون لدية خبر مهنية لمدة لا تقل عن 5 سنوات في نفس مجال تخصصه بعد تخرجه بالبكالوريوس في الهندسة من جامعة معترف بها بالإضافة إلى اجتياز اختبار FE.

هل اجتياز اختبار FE شرط لاختبار PE؟

نعم ، اجتياز اختبار FE شرط أساسي للتقديم لاختبار PE، ولا يوجد استثناء لذلك.

مرفق جميع تفاصيل الاختبارات المتاحة لدي الهيئة السعودية للمهندسين

مرفق شروط التسجيل والعضويات والترقية والدرجات المهنية لدي الهيئة السعودية للمهندسين

الجدول الزمني للدورة

الجدول الزمني للدورة

Week 1

28 July 2025	29 July 2025	30 July 2025	31 July 2025	1 August 2025	2 August 2025	3 August 2025
• Introduction • Mathematics Lecture	• Mathematics practice problems	• Mathematics practice problems	• Mathematics Lecture	• Mathematics practice problems	• Mathematics practice problems	• Mathematics Practice Problems Video

Week 2

4 August 2025	5 August 2025	6 August 2025	7 August 2025	8 August 2025	9 August 2025	10 August 2025
• Electrical materials properties Lecture	• Electrical materials properties Practice Problems	• Electrical materials properties Practice Problems	• Circuit analysis Lecture	• Circuit analysis Practice problems	• Circuit analysis Practice problems	• Circuit analysis • Electrical materials properties Practice Problems Video

Week 3

11 August 2025	12 August 2025	13 August 2025	14 August 2025	15 August 2025	16 August 2025	17 August 2025
• Linear systems Lecture	• Linear systems Practice problems	• Linear systems Practice problems	• Engineering economics Lecture	• Engineering economics Practice Problems	• Engineering economics Practice Problems	• Linear systems • Engineering economics Practice Problems Video

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الجدول الزمني للدورة

Week 4

18 August 2025	19 August 2025	20 August 2025	21 August 2025	22 August 2025	23 August 2025	24 August 2025
• Probability and statistics Lecture	• Probability and statistics Practice Problems	• Probability and statistics Practice Problems	• Power systems Lecture	• Power systems Practice problems	• Power systems Practice problems	• Probability and statistics • Power systems Practice Problems Video

Week 5

25 August 2025	26 August 2025	27 August 2025	28 August 2025	29 August 2025	30 August 2025	31 August 2025
• Control systems Lecture	• Control systems Practice Problems	• Control systems Practice Problems	• Digital systems Lecture	• Digital systems Practice Practice Problems	• Digital systems Practice Practice Problems	• Control systems • Digital systems Practice Problems Video

Week 6

1 September 2025	2 September 2025	3 September 2025	4 September 2025	5 September 2025	6 September 2025	7 September 2025
• Electronics Lecture	• Electronics Practice Problems	• Electronics Practice Problems	• Signal Processing • Communication Lecture	• Signal Processing • Communication Practice Problems	• Signal Processing • Communication Practice Problems	• Electronics • Signal Processing • Communication Practice Problems Video

الجدول الزمنى للدورة

Week 7

8 September 2025	9 September 2025	10 September 2025	11 September 2025	12 September 2025	13 September 2025	14 September 2025
• Computer Networks • Computer Systems Lecture	• Computer Networks • Computer Systems Practice Problems	• Computer Networks • Computer Systems Practice Problems	• Software engineering • Computer Networks Lecture	• Software engineering • Computer Networks Practice problems	• Software engineering • Computer Networks Practice problems	• Software engineering • Computer Networks • Computer Systems Practice Problems Video

Week 8

15 September 2025	16 September 2025	17 September 2025	18 September 2025	19 September 2025	20 September 2025	21 September 2025
• Electromagnetic Lecture	• Electromagnetic Practice Problems	• Electromagnetic Practice Problems	• Ethics and Professional Practice Lecture	• Ethics and Professional Practice Practice Problems	• Ethics and Professional Practice Practice Problems	• Electromagnetic • Ethics and Professional Practice Practice Problems Video

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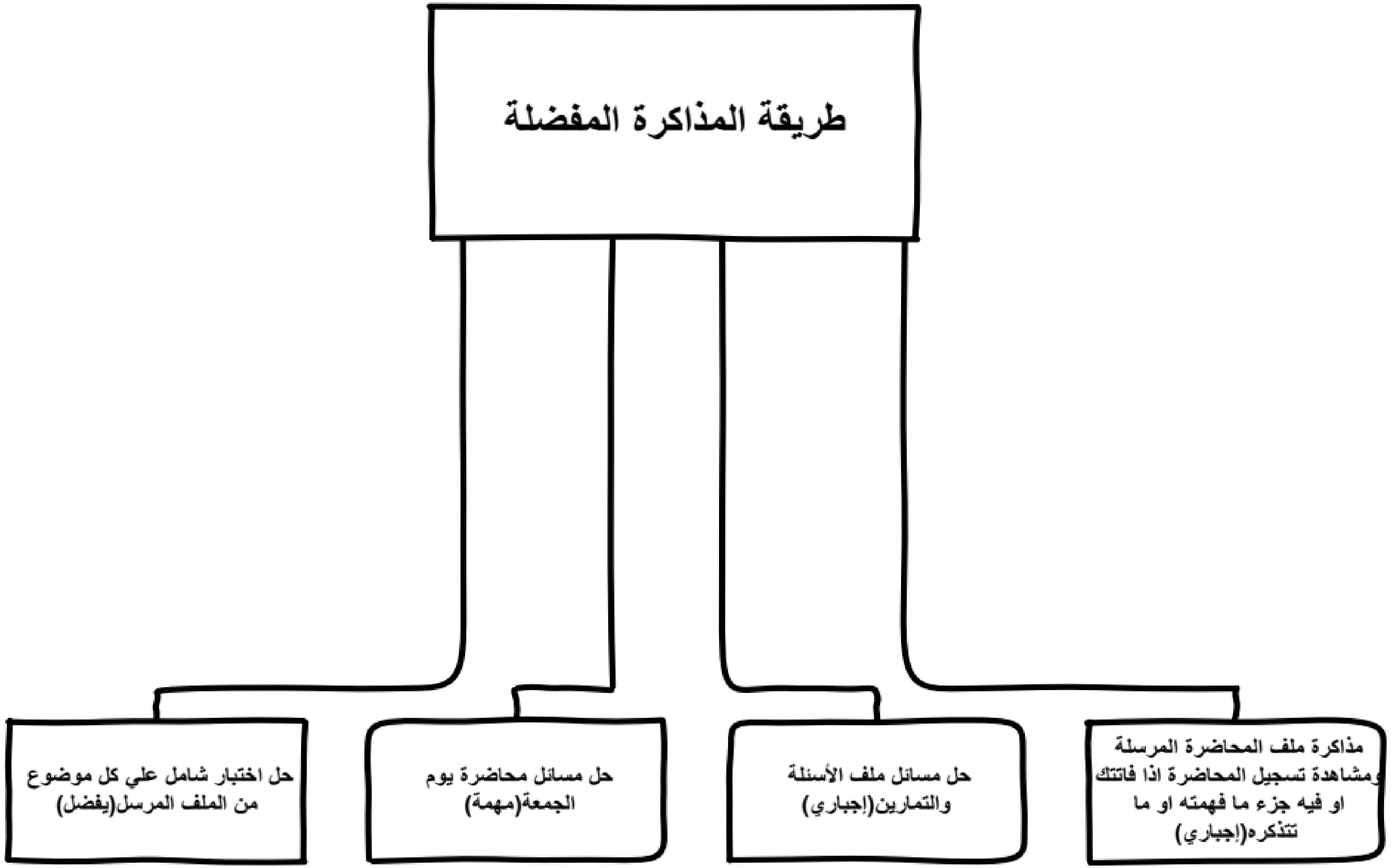
طريقة مذاكرة الدورة

طريقة مذاكرة الدورة



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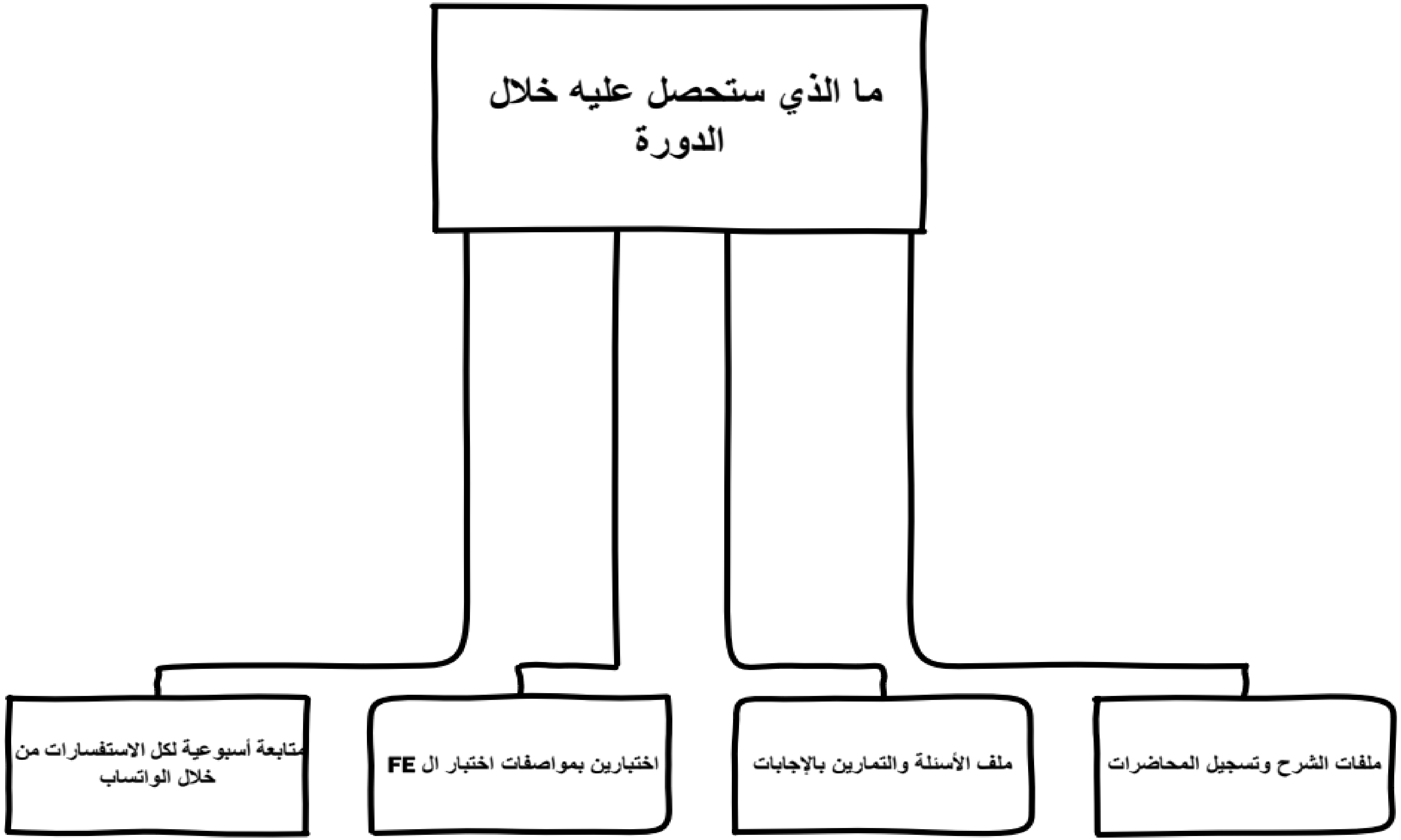
مرفقات الدورة

مرفقات الدورة



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Exam Structure and format

Exam Structure and format

Computer-Based Testing (CBT)

The FE Electrical Exam is administered year-round at testing centers worldwide.

You'll use a computer with specialized software to answer multiple-choice questions.

Exam Format

The exam consists of 110 multiple-choice questions.

You have 5 hours and 25 minutes to complete the exam, including a non-disclosure agreement, a tutorial, and a break.

Reference Handbook

NCEES HANDBOOK ED 10.5

Exam Structure and format

NCEES advancing licensure for engineers and surveyors

Fundamentals of Engineering (FE)
ELECTRICAL AND COMPUTER CBT Exam Specifications

Effective Beginning with the July 2020 Examinations

- The FE exam is a computer-based test (CBT). It is closed book with an electronic reference.
- Examinees have 6 hours to complete the exam, which contains 110 questions. The 6-hour time also includes a tutorial and an optional scheduled break.
- The FE exam uses both the International System of Units (SI) and the U.S. Customary System (USCS).

Knowledge	Number of Questions
1. Mathematics	**11–17**
A. Algebra and trigonometry	
B. Complex numbers	
C. Discrete mathematics	
D. Analytic geometry	
E. Calculus (e.g., differential, integral, single-variable, multivariable)	
F. Ordinary differential equations	
G. Linear algebra	
H. Vector analysis	
2. Probability and Statistics	**4–6**
A. Measures of central tendencies and dispersions (e.g., mean, mode, standard deviation)	
B. Probability distributions (e.g., discrete, continuous, normal, binomial, conditional probability)	
C. Expected value (weighted average)	
3. Ethics and Professional Practice	**4–6**
A. Codes of ethics (e.g., professional and technical societies, NCEES *Model Law* and *Model Rules*)	
B. Intellectual property (e.g., copyright, trade secrets, patents, trademarks)	
C. Safety (e.g., grounding, material safety data, PPE, radiation protection)	
4. Engineering Economics	**5–8**
A. Time value of money (e.g., present value, future value, annuities)	
B. Cost estimation	
C. Risk identification	
D. Analysis (e.g., cost-benefit, trade-off, breakeven)	
5. Properties of Electrical Materials	**4–6**
A. Semiconductor materials (e.g., tunneling, diffusion/drift current, energy bands, doping bands, p-n theory)	
B. Electrical (e.g., conductivity, resistivity, permittivity, magnetic permeability, noise)	
C. Thermal (e.g., conductivity, expansion)	
6. Circuit Analysis (DC and AC Steady State)	**11–17**
A. KCL, KVL	
B. Series/parallel equivalent circuits	
C. Thevenin and Norton theorems	
D. Node and loop analysis	
E. Waveform analysis (e.g., RMS, average, frequency, phase, wavelength)	
F. Phasors	
G. Impedance	
7. Linear Systems	**5–8**
A. Frequency/transient response	
B. Resonance	
C. Laplace transforms	
D. Transfer functions	
8. Signal Processing	**5–8**
A. Sampling (e.g., aliasing, Nyquist theorem)	
B. Analog filters	
C. Digital filters (e.g., difference equations, Z-transforms)	
9. Electronics	**7–11**
A. Models, biasing, and performance of discrete devices (e.g., diodes, transistors, thyristors)	
B. Amplifiers (e.g., single-stage/common emitter, differential, biasing)	
C. Operational amplifiers (e.g., ideal, nonideal)	
D. Instrumentation (e.g., measurements, data acquisition, transducers)	
E. Power electronics (e.g., rectifiers, inverters, converters)	

Exam Structure and format

10. Power Systems **8–12**

A. Power theory (e.g., power factor, single and three phase, voltage regulation)
B. Transmission and distribution (e.g., real and reactive losses, efficiency, voltage drop, delta and wye connections)
C. Transformers (e.g., single-phase and three-phase connections, reflected impedance)
D. Motors and generators (e.g., synchronous, induction, dc)

11. Electromagnetics **4–6**

A. Electrostatics/magnetostatics (e.g., spatial relationships, vector analysis)
B. Electrodynamics (e.g., Maxwell equations, wave propagation)
C. Transmission lines (high frequency)

12. Control Systems **6–9**

A. Block diagrams (e.g. feedforward, feedback)
B. Bode plots
C. Closed-loop response, open-loop response, and stability
D. Controller performance (e.g., steady-state errors, settling time, overshoot)

13. Communications **5–8**

A. Basic modulation/demodulation concepts (e.g., AM, FM, PCM)
B. Fourier transforms/Fourier series
C. Multiplexing (e.g., time division, frequency division, code division)
D. Digital communications

14. Computer Networks **4–6**

A. Routing and switching
B. Network topologies (e.g., mesh, ring, star)
C. Network types (e.g., LAN, WAN, internet)
D. Network models (e.g., OSI, TCP/IP)
E. Network intrusion detection and prevention (e.g., firewalls, endpoint detection, network detection)
F. Security (e.g., port scanning, network vulnerability testing, web vulnerability testing, penetration testing, security triad)

15. Digital Systems **8–12**

A. Number systems
B. Boolean logic
C. Logic gates and circuits
D. Logic minimization (e.g., SOP, POS, Karnaugh maps)
E. Flip-flops and counters
F. Programmable logic devices and gate arrays
G. State machine design
H. Timing (e.g., diagrams, asynchronous inputs, race conditions and other hazards)

16. Computer Systems **5–8**

A. Microprocessors
B. Memory technology and systems
C. Interfacing

17. Software Engineering **4–6**

A. Algorithms (e.g., sorting, searching, complexity, big-O)
B. Data structures (e.g., lists, trees, vectors, structures, arrays)
C. Software implementation (e.g., iteration, conditionals, recursion, control flow, scripting, testing)

Key Subject Areas and Calculators

Key Subject Areas and Calculators

1. Mathematics
2. Circuit analysis
3. Linear systems
4. Power systems
5. Digital systems
6. Control Systems
7. Electronics
8. Properties of electrical materials
9. Engineering economics
10. Signal processing
11. Communications
12. Computer systems
13. Computer networks
14. Software engineering
15. Electromagnetics
16. Probability and statistics
17. Ethics and professional practice

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References

References

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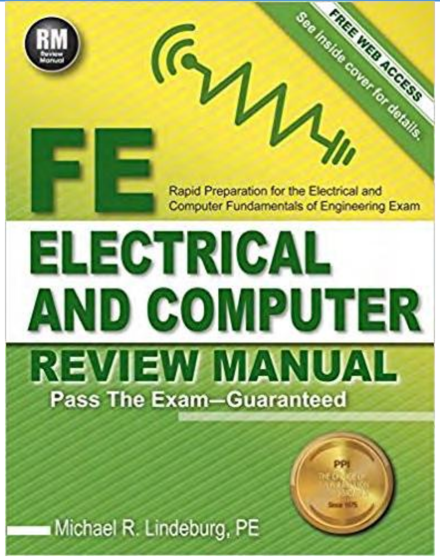

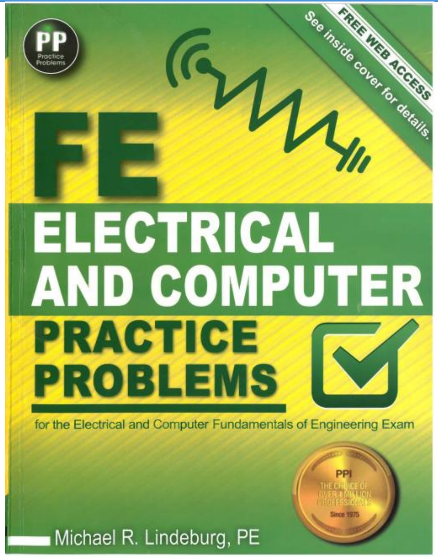

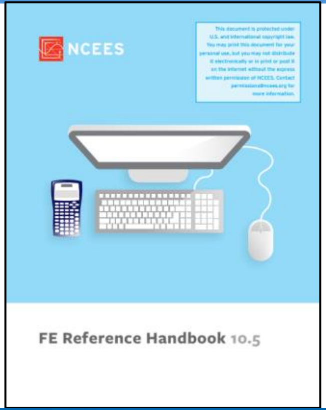

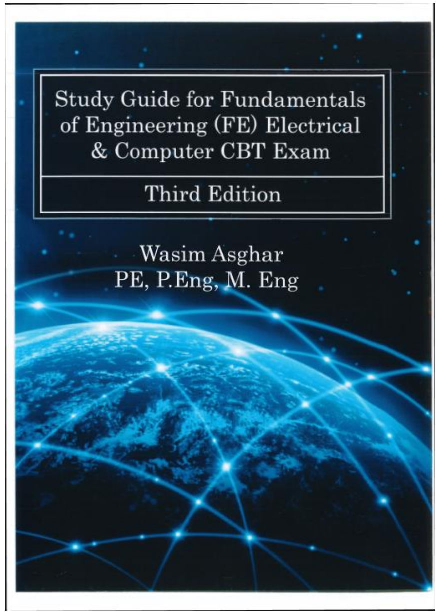

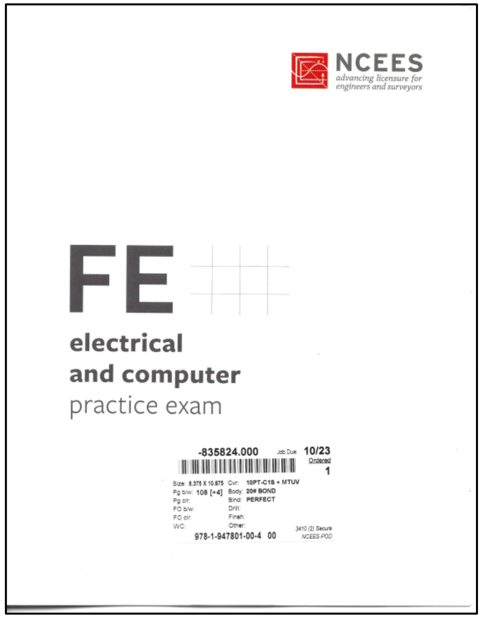

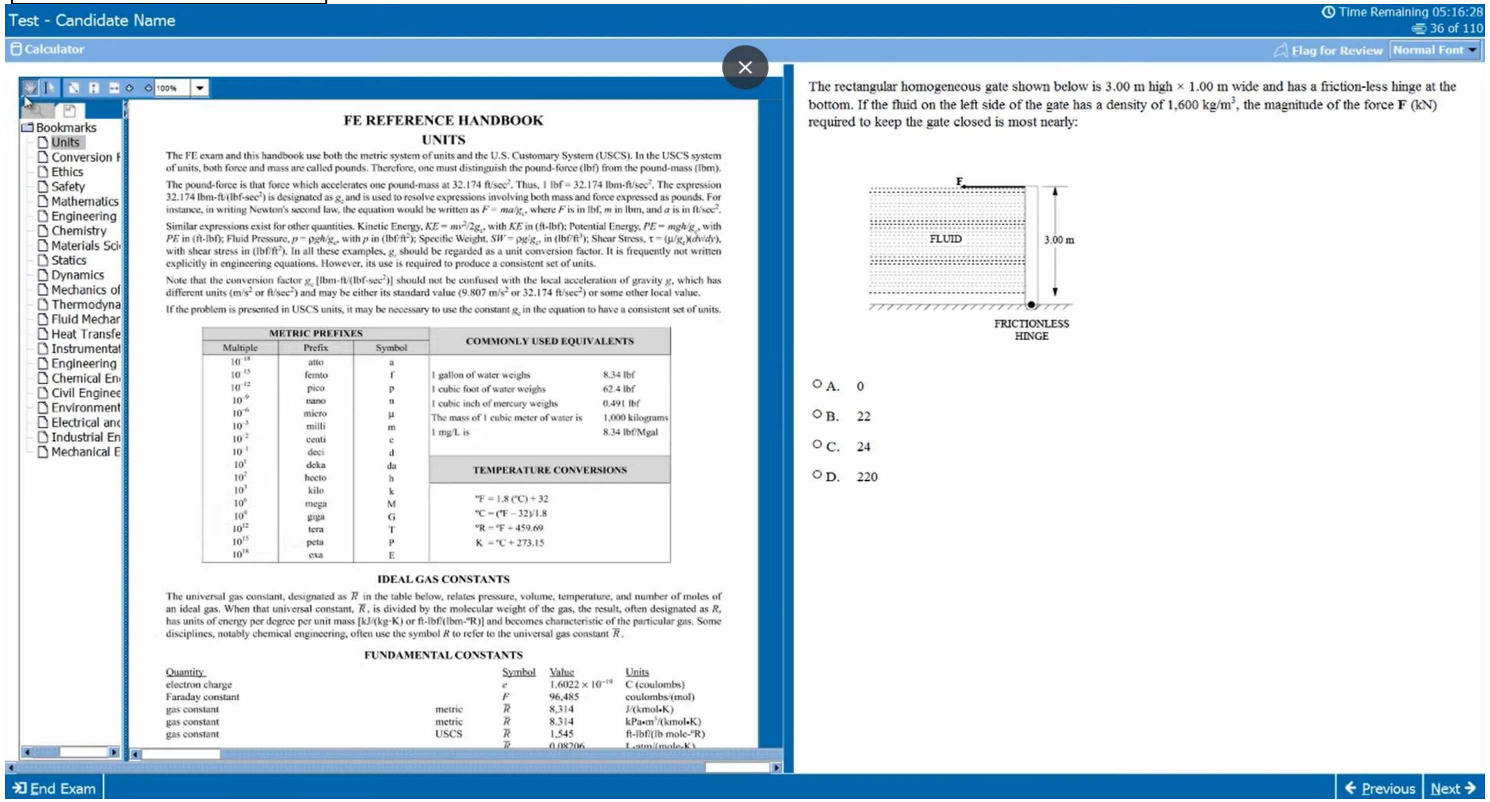

1

Mathematics

Key Topics

1.1

Complex numbers

Complex numbers

A complex number ***z*** can be written in rectangular form as

$z = x + jy$

$$j = \sqrt{-1}$$

x is the real part of ***z***
y is the imaginary part of ***z***.

The complex number *z* can also be written **in polar** or **exponential form** as

$$z = r\angle\phi = re^{j\phi}$$

where *r* is the magnitude of *z*.
and ϕ is the phase of *z*.

We notice that *z* can be represented in three ways:

$z = x + jy$	Rectangular form
$z = r\angle\phi$	Polar form
$z = re^{j\phi}$	Exponential form

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\phi, \qquad y = r\sin\phi$$

Complex numbers

the following operations are important.

Addition:

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$$

Subtraction:

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$$

Multiplication:

$$z_1 z_2 = r_1 r_2 \underline{/\phi_1 + \phi_2}$$

Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \underline{/\phi_1 - \phi_2}$$

Reciprocal:

$$\frac{1}{z} = \frac{1}{r} \underline{/-\phi}$$

Square Root:

$$\sqrt{z} = \sqrt{r} \underline{/\phi/2}$$

Complex Conjugate:

$$z^* = x - jy = r\underline{/-\phi} = re^{-j\phi}$$

$$\frac{1}{j} = -j$$

Problem 13. Evaluate in polar form:

(a) $\dfrac{16\angle 75°}{2\angle 15°}$

(a) $\dfrac{16\angle 75°}{2\angle 15°} = \dfrac{16}{2}\angle(75° - 15°) = \mathbf{8\angle 60°}$

$y = 3e^{-J\pi}$

$= 3\cos(-\pi \times \frac{180}{\pi}) + J\, 3\sin(-\pi \times \frac{180}{\pi})$

$= -3 + J0 = \boxed{-3}$

$y = 3e^{-J\frac{2}{3}\pi}$

$= 3\cos(-\frac{2}{3}\pi \times \frac{180}{\pi})$

$+ J\, 3\sin(-\frac{2}{3}\pi \times \frac{180}{\pi})$

$= -\frac{3}{2} - J\frac{3\sqrt{3}}{2} \rightarrow \frac{9}{2\sqrt{3}}$

Problem 4. Given $Z_1 = 2 + j4$ and $Z_2 = 3 - j$ determine (a) $Z_1 + Z_2$, (b) $Z_1 - Z_2$, (c) $Z_2 - Z_1$ and show the results on an Argand diagram

(a) $Z_1 + Z_2 = (2 + j4) + (3 - j)$

$= (2 + 3) + j(4 - 1) = \mathbf{5 + j3}$

(b) $Z_1 - Z_2 = (2 + j4) - (3 - j)$

$= (2 - 3) + j(4 - (-1)) = \mathbf{-1 + j5}$

(c) $Z_2 - Z_1 = (3 - j) - (2 + j4)$

$= (3 - 2) + j(-1 - 4) = \mathbf{1 - j5}$

Problem 12. Determine, in polar form:

(a) $8\angle 25° \times 4\angle 60°$

(b) $3\angle 16° \times 5\angle -44° \times 2\angle 80°$

(a) $8\angle 25° \times 4\angle 60° = (8 \times 4)\angle(25° + 60°) = \mathbf{32\angle 85°}$

(b) $3\angle 16° \times 5\angle -44° \times 2\angle 80°$

$= (3 \times 5 \times 2)\angle[16° + (-44°) + 80°] = \mathbf{30\angle 52°}$

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1.2

Vectors

Vectors

- Quantities such as time, temperature and mass are entirely defined by a numerical value and are called **scalars** or **scalar quantities**.
- The time taken to fill a water tank may be measured as, say, 50 s.
- Similarly, the temperature in a room may be measured as, say, 16∘C

- Quantities such as velocity, force and acceleration, which have both a **magnitude** and a **direction**, are **called vectors.**

- For example, the velocity of a car may be 90 km/h due West.
- a force of 20 N may act vertically downwards.
- or an acceleration of 10 m/s2 may act at 50∘ to the horizontal.

Components of a Vector

A vector A can be written in terms of unit vectors and its components.

$$\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

magnitude (length) can be calculated as $|\mathbf{A}| = L_{\mathbf{A}} = \sqrt{a_x^2 + a_y^2 + a_z^2}$

Example

Find the unit vector (i.e., the direction vector) associated with the origin-based vector $18\mathbf{i} + 3\mathbf{j} + 29\mathbf{k}$.

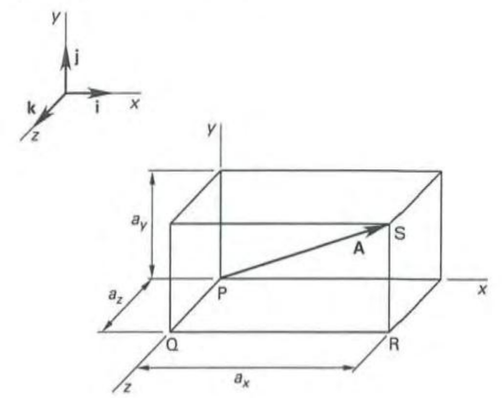
(A) $0.525\mathbf{i} + 0.088\mathbf{j} + 0.846\mathbf{k}$

(B) $0.892\mathbf{i} + 0.178\mathbf{j} + 0.416\mathbf{k}$

(C) $1.342\mathbf{i} + 0.868\mathbf{j} + 2.437\mathbf{k}$

(D) $6\mathbf{i} + \mathbf{j} + \frac{29}{3}\mathbf{k}$

Figure 3.1 Components of a Vector



Solution

The unit vector of a particular vector is the vector itself divided by its length.

$$\text{unit vector} = \frac{18\mathbf{i} + 3\mathbf{j} + 29\mathbf{k}}{\sqrt{(18)^2 + (3)^2 + (29)^2}} = 0.525\mathbf{i} + 0.088\mathbf{j} + 0.846\mathbf{k}$$

The answer is (A).

Vectors

❑ Vector Addition and Subtraction

$$\mathbf{A}+\mathbf{B}=(a_x+b_x)\mathbf{i}+(a_y+b_y)\mathbf{j}+(a_z+b_z)\mathbf{k}$$
$$\mathbf{A}-\mathbf{B}=(a_x-b_x)\mathbf{i}+(a_y-b_y)\mathbf{j}+(a_z-b_z)\mathbf{k}$$

Example

What is the sum of the two vectors $5\mathbf{i}+3\mathbf{j}-7\mathbf{k}$ and $10\mathbf{i}-12\mathbf{j}+5\mathbf{k}$?

(A) $8\mathbf{i}-7\mathbf{j}-\mathbf{k}$

(B) $10\mathbf{i}-9\mathbf{j}+3\mathbf{k}$

(C) $15\mathbf{i}-9\mathbf{j}-2\mathbf{k}$

(D) $15\mathbf{i}+7\mathbf{j}-3\mathbf{k}$

Solution

Use Eq. 3.2.

$$\begin{aligned}\mathbf{A}+\mathbf{B}&=(a_x+b_x)\mathbf{i}+(a_y+b_y)\mathbf{j}+(a_z+b_z)\mathbf{k}\\&=(5+10)\mathbf{i}+(3+(-12))\mathbf{j}+((-7)+5)\mathbf{k}\\&=15\mathbf{i}-9\mathbf{j}-2\mathbf{k}\end{aligned}$$

The answer is (C).

❑ Vector Dot Product

$$\mathbf{A}\cdot\mathbf{B}=a_xb_x+a_yb_y+a_zb_z$$
$$\mathbf{A}\cdot\mathbf{B}=|\mathbf{A}||\mathbf{B}|\cos\theta=\mathbf{B}\cdot\mathbf{A}$$

$$\begin{aligned}\theta&=\arccos\left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right)\\&=\arccos\left(\frac{a_xb_x+a_yb_y+a_zb_z}{|\mathbf{A}||\mathbf{B}|}\right)\end{aligned}$$

Example

What is the dot product, $\mathbf{A}\cdot\mathbf{B}$, of the vectors $\mathbf{A}=2\mathbf{i}+4\mathbf{j}+8\mathbf{k}$ and $\mathbf{B}=-2\mathbf{i}+\mathbf{j}-4\mathbf{k}$?

(A) $-4\mathbf{i}+4\mathbf{j}-32\mathbf{k}$

(B) $-4\mathbf{i}-4\mathbf{j}-32\mathbf{k}$

(C) -40

(D) -32

Solution

Use Eq. 3.4.

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B}&=a_xb_x+a_yb_y+a_zb_z\\&=(2)(-2)+(4)(1)+(8)(-4)\\&=-32\end{aligned}$$

The answer is (D).

❑ Vector Cross Product

$$\mathbf{A}\times\mathbf{B}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_x&a_y&a_z\\b_x&b_y&b_z\end{vmatrix}=-\mathbf{B}\times\mathbf{A}$$
$$\mathbf{A}\times\mathbf{B}=|\mathbf{A}||\mathbf{B}|\mathbf{n}\sin\theta$$

Example

What is the cross product, $\mathbf{A}\times\mathbf{B}$, of vectors $\mathbf{A}$ and $\mathbf{B}$?

$$\mathbf{A}=\mathbf{i}+4\mathbf{j}+6\mathbf{k}$$
$$\mathbf{B}=2\mathbf{i}+3\mathbf{j}+5\mathbf{k}$$

(A) $\mathbf{i}-\mathbf{j}-\mathbf{k}$

(B) $-\mathbf{i}+\mathbf{j}+\mathbf{k}$

(C) $2\mathbf{i}+7\mathbf{j}-5\mathbf{k}$

(D) $2\mathbf{i}+7\mathbf{j}+5\mathbf{k}$

Solution

Use Eq. 3.6. The cross product of two vectors is the determinant of a third-order matrix as shown.

$$\begin{aligned}\mathbf{A}\times\mathbf{B}&=\begin{bmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\a_x&a_y&a_z\\b_x&b_y&b_z\end{bmatrix}=\begin{bmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&4&6\\2&3&5\end{bmatrix}\\&=\mathbf{i}[(4)(5)-(6)(3)]-\mathbf{j}[(1)(5)-(6)(2)]\\&\quad+\mathbf{k}[(1)(3)-(4)(2)]\\&=2\mathbf{i}+7\mathbf{j}-5\mathbf{k}\end{aligned}$$

The answer is (C).

Vectors

Vector Dot Product identities

$$\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{A}$$
$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C})=\mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}$$
$$\mathbf{A}\cdot\mathbf{A}=|\mathbf{A}|^2$$

$$\mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1$$

$$\mathbf{i}\cdot\mathbf{j}=\mathbf{j}\cdot\mathbf{k}=\mathbf{k}\cdot\mathbf{i}=0$$

Example

What is the dot product $\mathbf{A}\cdot\mathbf{B}$ of unit vectors $\mathbf{A}=3\mathbf{i}$ and $\mathbf{B}=2\mathbf{i}$?

(A) –6

(B) –5

(C) 5

(D) 6

Solution

Use Eq. 3.11.

$$\mathbf{A}\cdot\mathbf{B}=3\mathbf{i}\cdot 2\mathbf{i}=(3\cdot 2)\mathbf{i}\cdot\mathbf{i}=(6)(1)$$
$$=6$$

The answer is (D).

Vector Cross Product identities

$$\mathbf{A}\times\mathbf{B}=-\mathbf{B}\times\mathbf{A}$$

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C})=(\mathbf{A}\times\mathbf{B})+(\mathbf{A}\times\mathbf{C})$$

$$(\mathbf{B}+\mathbf{C})\times\mathbf{A}=(\mathbf{B}\times\mathbf{A})+(\mathbf{C}\times\mathbf{A})$$

$$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0$$

$$\mathbf{i}\times\mathbf{j}=\mathbf{k}=-\mathbf{j}\times\mathbf{i}$$

$$\mathbf{k}\times\mathbf{i}=\mathbf{j}=-\mathbf{i}\times\mathbf{k}$$

Thank you

1.3

Algebra

Algebra

❖ LOGARITHMS:

$$\log_b(x) = c \quad [b^c = x]$$
$$\ln x \quad [\text{base} = e]$$
$$\log x \quad [\text{base} = 10]$$

Example

What is the value of $\log_{10} 1000$?

(A) 2

(B) 3

(C) 8

(D) 10

Solution

$\log_{10} 1000$ is the power of 10 that produces 1000. Use Eq. 2.1.

$$\log_b(x) = c \quad [b^c = x]$$
$$\log_{10} 1000 = c$$
$$10^c = 1000$$
$$c = 3$$

The answer is (B).

❖ Logarithmic Identities

$$\log_b b^n = n$$
$$\log x^c = c \log x$$
$$x^c = \text{antilog}(c \log x)$$
$$\log xy = \log x + \log y$$
$$\log_b b = 1$$
$$\log 1 = 0$$
$$\log x/y = \log x - \log y$$

Example

Which of the following is equal to $(0.001)^{2/3}$?

(A) $\text{antilog}\left(\frac{3}{2}\log 0.001\right)$

(B) $\frac{2}{3}\text{antilog}(\log 0.001)$

(C) $\text{antilog}\left(\log \frac{0.001}{\frac{2}{3}}\right)$

(D) $\text{antilog}\left(\frac{2}{3}\log 0.001\right)$

Solution

Use Eq. 2.5 and Eq. 2.6.

$$\log x^c = c \log x$$
$$\log (0.001)^{2/3} = \tfrac{2}{3}\log 0.001$$
$$(0.001)^{2/3} = \text{antilog}\left(\tfrac{2}{3}\log 0.001\right)$$

The answer is (D).

❖ Changing the Base

$$\log_b x = (\log_a x)/(\log_a b)$$

Example

Given that $\log_{10} 5 = 0.6990$ and $\log_{10} 9 = 0.9542$, what is the value of $\log_5 9$?

(A) 0.2550

(B) 0.7330

(C) 1.127

(D) 1.365

Solution

Use Eq. 2.11.

$$\log_b x = (\log_a x)/(\log_a b)$$
$$\log_5 9 = \frac{\log_{10} 9}{\log_{10} 5} = \frac{0.9542}{0.6990}$$
$$= 1.365$$

The answer is (D).

❖ POLYNOMIAL FUNCTIONS:

Quadratic Equations

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example

What are the roots of the quadratic equation $-7x + x^2 = -10$?

(A) –5 and 2

(B) –2 and 0.4

(C) 0.4 and 2

(D) 2 and 5

Solution

Rearrange the equation into the form of Eq. 2.24.

$$x^2 + (-7x) + 10 = 0$$

Use the quadratic formula, Eq. 2.25, with $a = 1$, $b = -7$, and $c = 10$.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-7) \pm \sqrt{(-7)^2 - (4)(1)(10)}}{(2)(1)} \\ &= 2 \text{ and } 5 \end{aligned}$$

The answer is (D).

1.4

Linear Algebra

Linear Algebra

❖ MATRICES

Addition of Matrices: Addition and subtraction of two matrices are possible only if both matrices have the **same number of rows and columns**.

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} + \begin{bmatrix} G & H & I \\ J & K & L \end{bmatrix} = \begin{bmatrix} A+G & B+H & C+I \\ D+J & E+K & F+L \end{bmatrix}$$

Problem 1. Add the matrices

(a) $\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$

(a) Adding the corresponding elements gives:

$$\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} = \begin{pmatrix} 2+(-3) & -1+0 \\ -7+7 & 4+(-4) \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$$

Multiplication of Matrices

when multiplying a matrix of dimensions (m by **n**), by a matrix of dimensions (**n** by r), the resulting matrix has dimensions **(m by r).** Thus a 2 by **3** matrix multiplied by a **3** by 1 matrix gives a matrix of dimensions **2 by 1**.

$$C = \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \cdot \begin{bmatrix} H & I \\ J & K \end{bmatrix} = \begin{bmatrix} (A \cdot H + B \cdot J) & (A \cdot I + B \cdot K) \\ (C \cdot H + D \cdot J) & (C \cdot I + D \cdot K) \\ (E \cdot H + F \cdot J) & (E \cdot I + F \cdot K) \end{bmatrix}$$

Example

What is the matrix product **AB** of matrices **A** and **B**?

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

(A) $\begin{bmatrix} 10 & 4 \\ 7 & 3 \end{bmatrix}$

(B) $\begin{bmatrix} 11 & 4 \\ 5 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} 8 & 3 \\ 2 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 10 & 7 \\ 4 & 3 \end{bmatrix}$

Solution

Use Eq. 6.2. Multiply the elements of each row in matrix **A** by the elements of the corresponding column in matrix **B**.

$$C = \begin{bmatrix} A & B \\ C & D \\ E & F \end{bmatrix} \cdot \begin{bmatrix} H & I \\ J & K \end{bmatrix} = \begin{bmatrix} (A \cdot H + B \cdot J) & (A \cdot I + B \cdot K) \\ (C \cdot H + D \cdot J) & (C \cdot I + D \cdot K) \\ (E \cdot H + F \cdot J) & (E \cdot I + F \cdot K) \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + 1 \times 2 & 2 \times 3 + 1 \times 1 \\ 1 \times 4 + 0 \times 2 & 1 \times 3 + 0 \times 1 \end{bmatrix} = \begin{bmatrix} 10 & 7 \\ 4 & 3 \end{bmatrix}$$

The answer is (D).

Transposes of Matrices

$$\mathbf{A} = \begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} A & D \\ B & E \\ C & F \end{bmatrix}$$

Example

What is the transpose of matrix **A**?

$$\mathbf{A} = \begin{bmatrix} 5 & 8 & 5 & 8 \\ 8 & 7 & 6 & 2 \end{bmatrix}$$

(A) $\begin{bmatrix} 8 & 7 & 6 & 2 \\ 5 & 8 & 5 & 8 \end{bmatrix}$

(B) $\begin{bmatrix} 2 & 6 & 7 & 8 \\ 8 & 5 & 8 & 5 \end{bmatrix}$

(C) $\begin{bmatrix} 8 & 5 \\ 7 & 8 \\ 6 & 5 \\ 2 & 8 \end{bmatrix}$

(D) $\begin{bmatrix} 5 & 8 \\ 8 & 7 \\ 5 & 6 \\ 8 & 2 \end{bmatrix}$

Solution

The transpose of a matrix is constructed by taking the ith row and making it the ith column.

The answer is (D).

Linear Algebra

❖ MATRICES

Determinants of 2 x 2 Matrices

$$|\mathbf{A}| = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

- If **A** has a row or column of zeros, the determinant is zero.
- If **A** has two identical rows or columns, the determinant is zero.

Example

What is the determinant of matrix **A**?

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$$

(A) 0
(B) 15
(C) 14
(D) 26

Solution

From Eq. 6.4, for a square 2 × 2 matrix,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix} = 3 \times 4 - 6 \times 2 = 0$$

The answer is (A).

Determinants of 3 x 3 Matrices

$$\mathbf{A} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$|\mathbf{A}| = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

Example

For the following set of equations, what is the determinant of the coefficient matrix?

$$10x + 3y + 10z = 5$$
$$8x - 2y + 9z = 5$$
$$8x + y - 10z = 5$$

(A) 598
(B) 620
(C) 714
(D) 806

Solution

Calculate the determinant of the coefficient matrix.

$$\begin{aligned} |\mathbf{A}| &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 \\ &\quad - a_2 b_1 c_3 - a_1 b_3 c_2 \\ &= (10)(-2)(-10) + (3)(9)(8) + (10)(8)(1) \\ &\quad - (8)(-2)(10) - (1)(9)(10) \\ &\quad - (-10)(8)(3) \\ &= 806 \end{aligned}$$

Inverse of Matrices

Problem 19. Find the inverse of

$$\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$$

$$\text{Inverse} = \frac{\text{adjoint}}{\text{determinant}}$$

The matrix of cofactors is $\begin{pmatrix} -17 & 9 & 15 \\ 23 & -13 & -21 \\ 18 & -10 & -16 \end{pmatrix}$

The transpose of the matrix of cofactors (i.e. the adjoint) is $\begin{pmatrix} -17 & 23 & 18 \\ 9 & -13 & -10 \\ 15 & -21 & -16 \end{pmatrix}$

The determinant of $\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$

$$= 1(7-24) - 5(-21+12) - 2(18-3)$$

$$= -17 + 45 - 30 = -2$$

Hence the inverse of $\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$

$$= \frac{\begin{pmatrix} -17 & 23 & 18 \\ 9 & -13 & -10 \\ 15 & -21 & -16 \end{pmatrix}}{-2}$$

$$= \begin{pmatrix} \mathbf{8.5} & \mathbf{-11.5} & \mathbf{-9} \\ \mathbf{-4.5} & \mathbf{6.5} & \mathbf{5} \\ \mathbf{-7.5} & \mathbf{10.5} & \mathbf{8} \end{pmatrix}$$

SOLVING SIMULTANEOUS LINEAR EQUATIONS

Example

Using Cramer's rule, what values of x, y, and z will satisfy the following system of simultaneous equations?

$$2x + 3y - 4z = 1$$
$$3x - y - 2z = 4$$
$$4x - 7y - 6z = -7$$

(A) $x = 1,\ y = -4,\ z = -1$

(B) $x = 1,\ y = 3,\ z = 1$

(C) $x = 3,\ y = -2,\ z = 4$

(D) $x = 3,\ y = 1,\ z = 2$

Solution

The determinant of the coefficient matrix is

$$|\mathbf{A}| = \begin{vmatrix} 2 & 3 & -4 \\ 3 & -1 & -2 \\ 4 & -7 & -6 \end{vmatrix} = 82$$

The determinants of the substitutional matrices are

$$|\mathbf{A}_1| = \begin{vmatrix} 1 & 3 & -4 \\ 4 & -1 & -2 \\ -7 & -7 & -6 \end{vmatrix} = 246$$

$$|\mathbf{A}_2| = \begin{vmatrix} 2 & 1 & -4 \\ 3 & 4 & -2 \\ 4 & -7 & -6 \end{vmatrix} = 82$$

$$|\mathbf{A}_3| = \begin{vmatrix} 2 & 3 & 1 \\ 3 & -1 & 4 \\ 4 & -7 & -7 \end{vmatrix} = 164$$

The values of x, y, and z that will satisfy the linear equations are

$$x = \frac{246}{82} = 3$$

$$y = \frac{82}{82} = 1$$

$$z = \frac{164}{82} = 2$$

The answer is (D).

1.5

Calculus

Calculus

❖ DERIVATIVES

In most cases, it is possible to transform a continuous function, I(x1,x2., X3," .), of One or more independent variables into a derivative function. **In simple two-dimensional cases, the derivative can be interpreted as the slope (tangent or rate of change) of the curve described by the original function.**

First Derivative

$y' =$ the slope of the curve $f(x)$

$$y = f(x)$$

$$D_x y = dy/dx = y'$$

Variations

$$f'(x),\ \frac{df(x)}{dx},\ \mathrm{D}f(x),\ \mathrm{D}_x f(x)$$

Example

What is the slope of the curve $y = 10x^2 - 3x - 1$ when it crosses the positive part of the x-axis?

(A) 3/20
(B) 1/5
(C) 1/3
(D) 7

Solution

The curve crosses the x-axis when $y = 0$. At this point,

$$10x^2 - 3x - 1 = 0$$

Use the quadratic equation or complete the square to determine the two values of x where the curve crosses the x-axis.

$$\begin{aligned} x^2 - 0.3x &= 0.1 \\ (x - 0.15)^2 &= 0.1 + (0.15)^2 \\ x &= \pm 0.35 + 0.15 \\ &= -0.2,\ 0.5 \end{aligned}$$

Since x must be positive, $x = 0.5$. The slope of the function is the first derivative.

$$\begin{aligned} \frac{dy}{dx} &= 20x - 3 \\ &= (20)(0.5) - 3 \\ &= 7 \end{aligned}$$

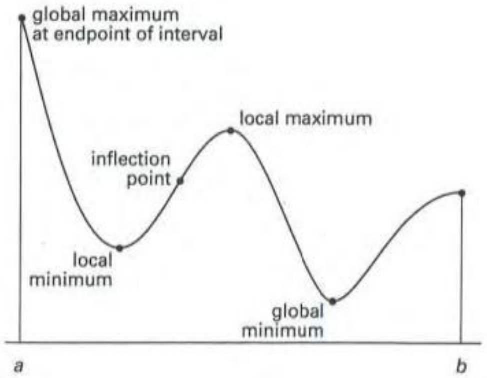
The answer is (D).

1. $dc/dx = 0$
2. $dx/dx = 1$
3. $d(cu)/dx = c\ du/dx$
4. $d(u + v - w)/dx = du/dx + dv/dx - dw/dx$
5. $d(uv)/dx = u\ dv/dx + v\ du/dx$
6. $d(uvw)/dx = uv\ dw/dx + uw\ dv/dx + vw\ du/dx$
7. $\dfrac{d(u/v)}{dx} = \dfrac{v\ du/dx - u\ dv/dx}{v^2}$
8. $d(u^n)/dx = nu^{n-1}\ du/dx$
9. $d[f(u)]/dx = \{d[f(u)]/du\}\ du/dx$
10. $du/dx = 1/(dx/du)$
11. $\dfrac{d(\log_a u)}{dx} = (\log_a e)\dfrac{1}{u}\dfrac{du}{dx}$
12. $\dfrac{d(\ln u)}{dx} = \dfrac{1}{u}\dfrac{du}{dx}$
13. $\dfrac{d(a^u)}{dx} = (\ln a)a^u\dfrac{du}{dx}$
14. $d(e^u)/dx = e^u\ du/dx$
15. $d(u^v)/dx = vu^{v-1}\ du/dx + (\ln u)\ u^v\ dv/dx$
16. $d(\sin u)/dx = \cos u\ du/dx$
17. $d(\cos u)/dx = -\sin u\ du/dx$
18. $d(\tan u)/dx = \sec^2 u\ du/dx$
19. $d(\cot u)/dx = -\csc^2 u\ du/dx$
20. $d(\sec u)/dx = \sec u \tan u\ du/dx$
21. $d(\csc u)/dx = -\csc u \cot u\ du/dx$
22. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ $\quad(-\pi/2 \le \sin^{-1} u \le \pi/2)$
23. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{1}{\sqrt{1-u^2}}\dfrac{du}{dx}$ $\quad(0 \le \cos^{-1} u \le \pi)$
24. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{1}{1+u^2}\dfrac{du}{dx}$ $\quad(-\pi/2 < \tan^{-1} u < \pi/2)$
25. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{1}{1+u^2}\dfrac{du}{dx}$ $\quad(0 < \cot^{-1} u < \pi)$
26. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$ $\quad(0 < \sec^{-1} u < \pi/2)(-\pi \le \sec^{-1} u < -\pi/2)$
27. $\dfrac{d(\csc^{-1} u)}{dx} = -\dfrac{1}{u\sqrt{u^2-1}}\dfrac{du}{dx}$ $\quad(0 < \csc^{-1} u \le \pi/2)(-\pi < \csc^{-1} u \le -\pi/2)$

❖ CRITICAL POINTS

Derivatives are used to locate the local critical points, that is, extreme points (also known as **maximum** and **minimum** points) as well as the **inflection points** (points of confluxes) of functions of one variable. **The plurals extrema, maxima, and minima are used without the word "points." These points are illustrated in Fig. 7.1. There is usually an inflection point between two adjacent local extrema.**

Figure 7.1 Critical Points



Test for a Maximum

$$f'(a) = 0$$

$$f''(a) < 0$$

Description

For a function $f(x)$ with an extreme point at $x = a$, if the point is a maximum, then the second derivative is negative.

Example

What is the maximum value of the function $f(x) = -x^2 - 8x + 1$?

(A) 1

(B) 4

(C) 8

(D) 17

Solution

Use Eq. 7.33 and Eq. 7.34.

$$f(x) = -x^2 - 8x + 1$$
$$f'(x) = -2x - 8$$
$$f''(x) = -2$$

$f'(x) = 0$ when x is equal to -4, and $f''(x)$ is less than zero, so $f(x)$ has its maximum value at $x = -4$.

$$\begin{aligned} f(x) &= -x^2 - 8x + 1 \\ &= -(-4)^2 - (8)(-4) + 1 \\ &= 17 \end{aligned}$$

The answer is (D).

Calculus

Test for a Minimum

$f'(a) = 0$

$f''(a) > 0$

Description

For a function $f(x)$ with a critical point at $x = a$, if the point is a minimum, then the second derivative is positive.

Example

What is the minimum value of the function $f(x) = 3x^2 + 3x - 5$?

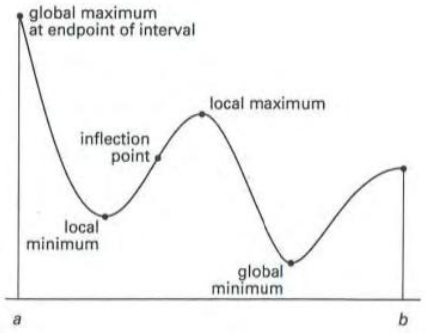
(A) -12.0

(B) -8.0

(C) -5.75

(D) -5.00

Figure 7.1 *Critical Points*



Solution

Use Eq. 7.35 and Eq. 7.36.

$$f(x) = 3x^2 + 3x - 5$$
$$f'(x) = 6x + 3$$
$$f''(x) = 6$$

$f'(x) = 0$ when x is equal to -0.5, and $f''(x)$ is greater than zero, so $f(x)$ has its minimum value at $x = -0.5$.

$$\begin{aligned} f(x) &= 3x^2 + 3x - 5 \\ &= (3)(-0.5)^2 + (3)(-0.5) - 5 \\ &= -5.75 \end{aligned}$$

The answer is (C).

Test for a Point of Inflection

$f''(a) = 0$

Description

For a function $f(x)$ with $f'(x) = 0$ at $x = a$, if the point is a point of inflection, then Eq. 7.37 is true.

Problem 1.5 g) Calculate the local minimum and maximum points of function given below.

$$f(x) = 3x^3 + 3x^2 - 3x + 3 \quad -2 \le x \le 1$$

(A) There is no minimum/maximum

(B) $x = \frac{1}{3}$ (min), $x = -1$ (max)

(C) $x = \frac{1}{3}$ (max), $x = -1$ (min)

(D) $x = \frac{1}{3}$ (min), no max

Problem 1.5 h) The point of inflection for function given in problem 1.5 g) exists at $x =$ _______.

1.5 g) CORRECT ANSWER - B

According to the problem statement:

$f(x) = 3x^3 + 3x^2 - 3x + 3 \qquad -2 \le x \le 1$

To calculate the maximum and minimum of a function, first order derivative is calculated as shown below:

$f'(x) = 9x^2 + 6x - 3$

The roots of $f'(x)$ can be calculated using factorization, quadratic equation, or calculator.

$9x^2 + 6x - 3 = 0 \rightarrow x = \frac{1}{3}, x = -1$

Now, we need to calculate second order derivative $f''(x) = 18x + 6$

Substituting $x = -1$ in second derivative equation results in negative value. Therefore, it is a local maximum.

Substituting $x = 1/3$ in second derivative equation results in positive value. Therefore, it is a local minimum.

1.5 h) CORRECT ANSWER: $x = -1/3$

In the previous problem, $f''(x) = 18x + 6$

Point of inflection is calculated by taking second order derivative and setting it to zero as shown below:

$f''(x) = 18x + 6 = 0 \rightarrow x = -1/3$

The calculated value of x is a point of inflection because at this point second derivative is zero and second derivative also changes sign as x increases through it (test by substituting incremental values).

PARTIAL DERIVATIVES

$$z = f(x, y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial f(x, y)}{\partial x}$$

Variations

Symbols for a partial derivative of $f(x, y)$ taken with respect to variable x are $\partial f/\partial x$ and $f_x(x, y)$.

Example

What is the partial derivative with respect to x of the following function?

$$z = e^{xy}$$

(A) e^{xy}

(B) $\frac{e^{xy}}{x}$

(C) $\frac{e^{xy}}{y}$

(D) ye^{xy}

Solution

Use Eq. 7.19 and Eq. 7.39. The partial derivative is

$$d(e^u)/dx = e^u du/dx$$

$$\frac{\partial z}{\partial x} = \frac{\partial e^{xy}}{\partial x} = e^{xy}\frac{\partial (xy)}{\partial x} = ye^{xy}$$

The answer is (D).

INTEGRALS

Integration is the inverse operation of differentiation. There are two types of integrals: **definite integrals**, which are restricted to a specific range of the independent variable, and **indefinite integrals**, which are unrestricted. Indefinite integrals are sometimes referred to as antiderivatives.

Example

What is the approximate total area bounded by $y = \sin x$ over the interval $0 \leq x \leq 2\pi$? (x is in radians.)

(A) 0

(B) $\pi/2$

(C) 2

(D) 4

Solution

The integral of $f(x)$ represents the area under the curve $f(x)$ between the limits of integration. However, since the value of $\sin x$ is negative in the range $\pi \leq x \leq 2\pi$, the total area would be calculated as zero if the integration was carried out in one step. The integral could be calculated over two ranges, but it is easier to exploit the symmetry of the sine curve.

$$\begin{aligned} A &= \int_{x_1}^{x_2} f(x)\,dx = \int_0^{2\pi} |\sin x|\,dx \\ &= 2\int_0^{\pi} \sin x\,dx \\ &= -2\cos x\Big|_0^{\pi} \\ &= (-2)(-1-1) \\ &= 4 \end{aligned}$$

The answer is (D).

❖ LIMITS

A *limit* is the value a function approaches when an independent variable approaches a target value. For example, suppose the value of $y = x^2$ is desired as x approaches 5. This could be written as

$$y(5) = \underset{x \to 5}{\text{limit}}\, x^2$$

general case of a limit evaluated as x approaches the target value a is written as

$$\lim_{x \to a} f(x)$$

❖ L'Hôpital's Rule

$$\underset{x \to \alpha}{\text{limit}} \frac{f'(x)}{g'(x)}, \quad \underset{x \to \alpha}{\text{limit}} \frac{f''(x)}{g''(x)}, \quad \underset{x \to \alpha}{\text{limit}} \frac{f'''(x)}{g'''(x)}$$

Description

L'Hôpital's rule may be used only when the numerator and denominator of the expression are both indeterminate (i.e., are both zero or are both infinite) at the limit point. $f^k(x)$ and $g^k(x)$ are the kth derivatives of the functions $f(x)$ and $g(x)$, respectively. L'Hôpital's rule

$\frac{0}{0}$, $\frac{\infty}{\infty}$

Example

Evaluate the following limit.

$$\underset{x \to 0}{\text{limit}} \frac{1 - e^{3x}}{4x}$$

(A) $-\infty$

(B) $-3/4$

(C) 0

(D) $1/4$

Solution

This limit has the indeterminate form 0/0, so use L'Hôpital's rule.

$$\underset{x \to \alpha}{\text{limit}} \frac{f(x)}{g(x)} = \underset{x \to \alpha}{\text{limit}} \frac{f'(x)}{g'(x)}$$

$$\underset{x \to 0}{\text{limit}} \frac{1 - e^{3x}}{4x} = \underset{x \to 0}{\text{limit}} \frac{-3e^{3x}}{4}$$

$$= -3/4$$

The answer is (B).

Evaluate:

$$\lim_{x \to \infty} \frac{x^3 - 2x + 9}{2x^3 - 8}$$

ANS: $\frac{1}{2}$

1.6

Trigonometry

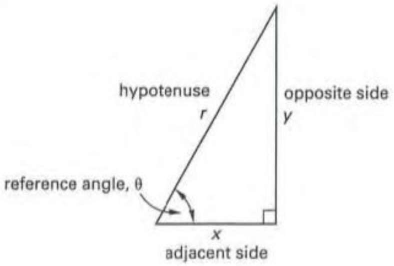
Trigonometry

❖ DEGREES AND RADIANS

multiply	by	to obtain
radians	$\frac{180}{\pi}$	degrees
degrees	$\frac{\pi}{180}$	radians

RIGHT TRIANGLES

Figure 5.5 Right Triangle

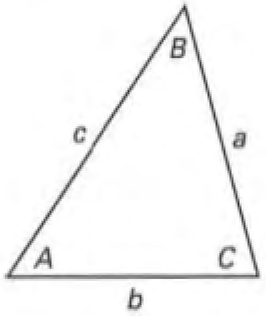


Trigonometric Functions

$$\sin\theta = y/r$$
$$\cos\theta = x/r$$
$$\tan\theta = y/x$$
$$\csc\theta = r/y$$
$$\sec\theta = r/x$$
$$\cot\theta = x/y$$

GENERAL TRIANGLES

Figure 5.7 General Triangle



Law of Sines

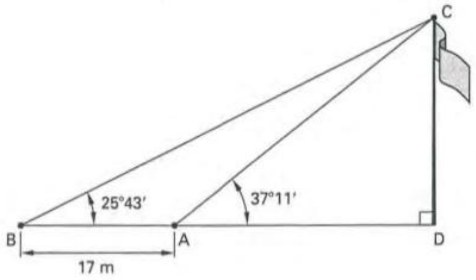
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Example

The vertical angle to the top of a flagpole from point A on the ground is observed to be 37° 11′. The observer walks 17 m directly away from the flagpole from point A to point B and finds the new angle to be 25° 43′.

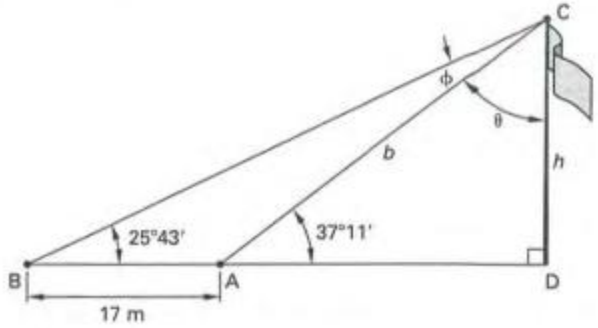


What is the approximate height of the flagpole?

(A) 10 m
(B) 22 m
(C) 82 m
(D) 300 m

Solution

The two observations lead to two triangles with a common leg, h.



Find angle θ in triangle ADC.

$$37°\,11' + 90° + \theta = 180°$$
$$\theta = 52°\,49'$$

Find angle ϕ in triangle BDC.

$$25°\,43' + 90° + (52°\,49' + \phi) = 180°$$
$$\phi = 11°\,28'$$

Use the law of sines on triangle BAC to find side b.

$$\frac{\sin 11°\,28'}{17\text{ m}} = \frac{\sin 25°\,43'}{b}$$
$$b = 37.11\text{ m}$$

Find the flagpole height, h, using triangle ADC.

$$\sin 37°\,11' = \frac{h}{b}$$
$$h = b\sin 37°\,11'$$
$$= (37.11\text{ m})\sin 37°\,11'$$
$$= 22.43\text{ m} \quad (22\text{ m})$$

The answer is (B).

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Trigonometry

❖ **TRIGONOMETRIC IDENTITIES**

$$\csc\theta = 1/\sin\theta$$
$$\sec\theta = 1/\cos\theta$$
$$\cot\theta = 1/\tan\theta$$

Example

Simplify the expression $\cos\theta\sec\theta/\tan\theta$.

(A) 1
(B) $\cot\theta$
(C) $\csc\theta$
(D) $\sin\theta$

Solution

Use the reciprocal functions given in Eq. 5.12 and Eq. 5.13.

$$\frac{\cos\theta\sec\theta}{\tan\theta} = \frac{\cos\theta\left(\frac{1}{\cos\theta}\right)}{\tan\theta} = \frac{1}{\tan\theta} = \cot\theta$$

The answer is (B).

$$\cos\theta = \sin(\theta+\pi/2) = -\sin(\theta-\pi/2)$$
$$\sin\theta = \cos(\theta-\pi/2) = -\cos(\theta+\pi/2)$$
$$\tan\theta = \sin\theta/\cos\theta$$
$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$\cot^2\theta + 1 = \csc^2\theta$$

Example

Which of the following expressions is equivalent to the expression $\csc\theta\cos^3\theta\tan\theta$?

(A) $\sin\theta$
(B) $\cos\theta$
(C) $1-\sin^2\theta$
(D) $1+\sin^2\theta$

Solution

Simplify the expression using the trigonometric identities given in Eq. 5.11, Eq. 5.16, and Eq. 5.17.

$$\csc\theta\cos^3\theta\tan\theta = \left(\frac{1}{\sin\theta}\right)\cos^3\theta\left(\frac{\sin\theta}{\cos\theta}\right) = \cos^2\theta = 1-\sin^2\theta$$

The answer is (C).

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \quad 5.20$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1-2\sin^2\alpha = 2\cos^2\alpha - 1 \quad 5.21$$
$$\tan 2\alpha = (2\tan\alpha)/(1-\tan^2\alpha) \quad 5.22$$
$$\cot 2\alpha = (\cot^2\alpha - 1)/(2\cot\alpha) \quad 5.23$$

Example

What is an equivalent expression for $\sin 2\alpha$?

(A) $-2\sin\alpha\cos\alpha$
(B) $\frac{1}{2}\sin\alpha\cos\alpha$
(C) $\dfrac{2\sin\alpha}{\sec\alpha}$
(D) $2\sin\alpha\cos\dfrac{\alpha}{2}$

Solution

Use Eq. 5.20, the double-angle formula for the sine function.

$$\sin 2\alpha = 2\sin\alpha\cos\alpha = \frac{2\sin\alpha}{\sec\alpha}$$

The answer is (C).

Trigonometry

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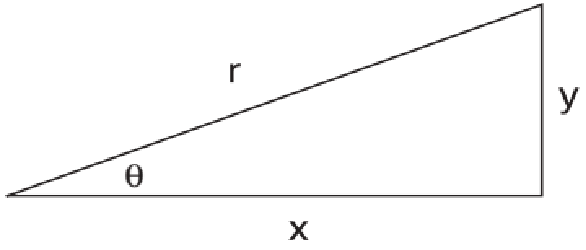
Trigonometry

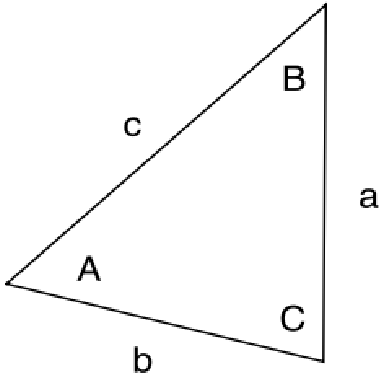
Trigonometric functions are defined using a right triangle.

$\sin \theta = y/r$, $\cos \theta = x/r$

$\tan \theta = y/x$, $\cot \theta = x/y$

$\csc \theta = r/y$, $\sec \theta = r/x$





Law of Sines $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Law of Cosines

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

Identities

$\cos \theta = \sin (\theta + \pi/2) = -\sin (\theta - \pi/2)$

$\sin \theta = \cos (\theta - \pi/2) = -\cos (\theta + \pi/2)$

$\csc \theta = 1/\sin \theta$

$\sec \theta = 1/\cos \theta$

$\tan \theta = \sin \theta/\cos \theta$

$\cot \theta = 1/\tan \theta$

$\sin^2 \theta + \cos^2 \theta = 1$

$\tan^2 \theta + 1 = \sec^2 \theta$

$\cot^2 \theta + 1 = \csc^2 \theta$

$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

$\sin 2\alpha = 2 \sin \alpha \cos \alpha$

$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha = 2 \cos^2 \alpha - 1$

$\tan 2\alpha = (2 \tan \alpha)/(1 - \tan^2 \alpha)$

$\cot 2\alpha = (\cot^2 \alpha - 1)/(2 \cot \alpha)$

$\tan (\alpha + \beta) = (\tan \alpha + \tan \beta)/(1 - \tan \alpha \tan \beta)$

$\cot (\alpha + \beta) = (\cot \alpha \cot \beta - 1)/(\cot \alpha + \cot \beta)$

$\sin (\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$

$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$

$\tan (\alpha - \beta) = (\tan \alpha - \tan \beta)/(1 + \tan \alpha \tan \beta)$

$\cot (\alpha - \beta) = (\cot \alpha \cot \beta + 1)/(\cot \beta - \cot \alpha)$

$\sin (\alpha/2) = \pm\sqrt{(1 - \cos \alpha)/2}$

$\cos (\alpha/2) = \pm\sqrt{(1 + \cos \alpha)/2}$

$\tan (\alpha/2) = \pm\sqrt{(1 - \cos \alpha)/(1 + \cos \alpha)}$

$\cot (\alpha/2) = \pm\sqrt{(1 + \cos \alpha)/(1 - \cos \alpha)}$

$\sin \alpha \sin \beta = (1/2)[\cos (\alpha - \beta) - \cos (\alpha + \beta)]$

$\cos \alpha \cos \beta = (1/2)[\cos (\alpha - \beta) + \cos (\alpha + \beta)]$

$\sin \alpha \cos \beta = (1/2)[\sin (\alpha + \beta) + \sin (\alpha - \beta)]$

$\sin \alpha + \sin \beta = 2 \sin [(1/2)(\alpha + \beta)] \cos [(1/2)(\alpha - \beta)]$

$\sin \alpha - \sin \beta = 2 \cos [(1/2)(\alpha + \beta)] \sin [(1/2)(\alpha - \beta)]$

$\cos \alpha + \cos \beta = 2 \cos [(1/2)(\alpha + \beta)] \cos [(1/2)(\alpha - \beta)]$

$\cos \alpha - \cos \beta = -2 \sin [(1/2)(\alpha + \beta)] \sin [(1/2)(\alpha - \beta)]$

1.7

Differential Equations

Differential Equations

A ***differential equation*** is a mathematical expression combining a function (e.g., *Y=f(x))* and one or more of its derivatives. **The *order* of a differential equation** is the **highest derivative in it**. *First-order differential equations* contain only first derivatives of the function, *second order differential equations* contain second derivatives (and may contain first derivatives as well).

Linear Differential Equation with Constant Coefficients

$$b_n \frac{d^n y(x)}{dx^n} + \cdots + b_1 \frac{dy(x)}{dx} + b_0 y(x) = f(x)$$

$[b_n, \ldots, b_i, \ldots, b_1, \text{and } b_0 \text{ are constants}]$

Example

Which of the following is NOT a linear differential equation?

(A) $5\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 16y = 4te^{-7t}$

(B) $5\frac{d^2y}{dt^2} - 8t^2\frac{dy}{dt} + 16y = 0$

(C) $5\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 16y = \frac{dy}{dy}$

(D) $5\left(\frac{dy}{dt}\right)^2 - 8\frac{dy}{dt} + 16y = 0$

Solution

A linear differential equation consists of multiples of a function, $y(t)$, and its derivatives, d^ny/dt^n. The multipliers may be scalar constants or functions, $g(t)$, of the independent variable, t. The forcing function, $f(t)$, (i.e., the right-hand side of the equation) may be 0, a constant, or any function of the independent variable, t. The multipliers cannot be higher powers of the function, $y(t)$.

The answer is (D).

❖ Homogeneous First Order Linear Differential Equations

$$y' + ay = 0 \qquad \frac{dy}{dt} + ay = 0$$

$$y = Ce^{-at} \qquad f(t) = Ce^{-at}$$

Example

Which of the following is the general solution to the differential equation and boundary conditions?

$$\frac{dy}{dt} - 5y = 0$$
$$y(0) = 3$$

(A) $-\frac{1}{3}e^{-5t}$

(B) $3e^{5t}$

(C) $5e^{-3t}$

(D) $\frac{1}{5}e^{-3t}$

Solution

This is a first-order, linear differential equation. The characteristic equation is $r - 5 = 0$. The root, r, is 5.

The solution is in the form of Eq. 8.5.

$$y = Ce^{5t}$$

The initial condition is used to find C.

$$y(0) = Ce^{(5)(0)} = 3$$
$$C = 3$$
$$y = 3e^{5t}$$

The answer is (B).

❖ **Homogeneous Second Order Linear Differential Equations**

$$y'' + ay' + by = 0$$

$$(r^2 + ar + b)Ce^{rx} = 0$$

$$r^2 + ar + b = 0$$

Roots of the Characteristic Equation

$$r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

$$y = (C_1 + C_2 x)e^{r_1 x}$$

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

$$\alpha = -a/2$$

$$\beta = \frac{\sqrt{4b - a^2}}{2}$$

If $a^2 > 4b$, the solution is of the form (overdamped)

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

If $a^2 = 4b$, the solution is of the form (critically damped)

$$y = (C_1 + C_2 x)e^{r_1 x}$$

If $a^2 < 4b$, the solution is of the form (underdamped)

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x), \text{ where}$$

$$\alpha = -a/2$$

$$\beta = \frac{\sqrt{4b - a^2}}{2}$$

Example

What is the general solution to the following homogeneous differential equation?

$$y'' - 8y' + 16y = 0$$

(A) $y = C_1 e^{4x}$

(B) $y = (C_1 + C_2 x)e^{4x}$

(C) $y = C_1 e^{-4x} + C_2 e^{4x}$

(D) $y = C_1 e^{2x} + C_2 e^{4x}$

Solution

Find the roots of the characteristic equation.

$$r^2 - 8r + 16 = 0$$

$$a = -8$$

$$b = 16$$

From Eq. 8.9,

$$r_{1,2} = r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$$= \frac{-(-8) \pm 2\sqrt{(-8)^2 - (4)(16)}}{2}$$

$$= 4, 4$$

Because $a^2 = 4b$, the characteristic equation has double roots, and the solution takes the form

$$y = (C_1 + C_2 x)e^{rx}$$

$$= (C_1 + C_2 x)e^{4x}$$

The answer is (B).

❖ **First-Order Linear Nonhomogeneous Differential Equations with Constant Coefficients**

$$\tau\frac{dy}{dt}+y=Kx(t) \qquad x(t)=\begin{Bmatrix} A & t<0 \\ B & t>0 \end{Bmatrix}$$

$$y(0)=KA$$

τ = time constant
K = gain

The solution is

$$y(t) = KA + (KB - KA)\left(1 - \exp\left(\frac{-t}{\tau}\right)\right) \text{ or}$$

$$\frac{t}{\tau} = \ln\left[\frac{KB - KA}{KB - y}\right]$$

Example

A spring-mass-dashpot system starting from a motionless state is acted upon by a step function. The response is described by the differential equation in which time, t, is given in seconds measured from the application of the ramp function.

$$\frac{dy}{dt} + 2y = 2u(0) \quad [y(0) = 0]$$

How long will it take for the system to reach 63% of its final value?

(A) 0.25 s

(B) 0.50 s

(C) 1.0 s

(D) 2.0 s

Solution

To fit this problem into the format used by Eq. 8.16, the coefficient of y must be 1. Dividing by 2,

$$0.5\frac{dy}{dt} + y = tu(0)$$

$$\tau = 0.50 \text{ s}$$

The answer is (B).

1.8

Analytic Geometry

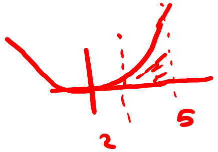
Analytic Geometry

❖ **Mensuration of Areas and Volumes**

Nomenclature

A = total surface area
P = perimeter
V = volume

Ex: If the equation of the area of parabola is given by $y = x^2$ And Two line intrupt x-axis in $x = 2$, $x = 5$ find the area

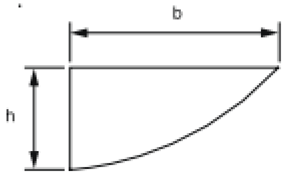


$$A = \int_2^5 y\,dx = \int_2^5 x^2\,dx = \frac{x^3}{3}\Big|_2^5$$

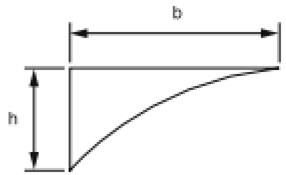
$$\frac{1}{3}(5^3 - 2^3) = \frac{1}{3} \times 117$$

$$= \frac{117}{3}$$

Parabola

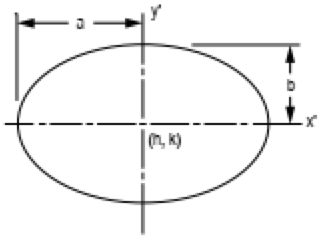


$A = 2bh/3$



$A = bh/3$

Ellipse



$A = \pi ab$

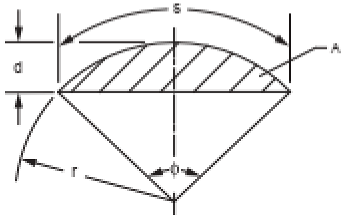
$P_{approx} = 2\pi\sqrt{(a^2 + b^2)/2}$

$$P = \pi(a+b)\left[\begin{array}{l} 1 + (1/2)^2\lambda^2 + (1/2 \times 1/4)^2\lambda^4 \\ +(1/2 \times 1/4 \times 3/6)^2\lambda^6 + (1/2 \times 1/4 \times 3/6 \times 5/8)^2\lambda^8 \\ +(1/2 \times 1/4 \times 3/6 \times 5/8 \times 7/10)^2\lambda^{10} + \ldots \end{array}\right]$$

where

$\lambda = (a - b)/(a + b)$

Circular Segment

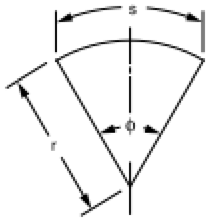


$A = [r^2(\phi - \sin\phi)]/2$

$\phi = s/r = 2\{\arccos[(r - d)/r]\}$

Source: Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

Circular Sector

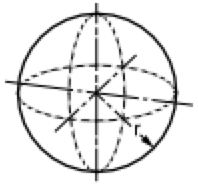


$A = \phi r^2/2 = sr/2$

$\phi = s/r$

Source: Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

Sphere



$V = 4\pi r^3/3 = \pi d^3/6$

$A = 4\pi r^2 = \pi d^2$

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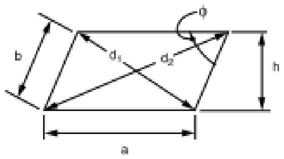
❖ **Mensuration of Areas and Volumes**

Nomenclature

A = total surface area
P = perimeter
V = volume

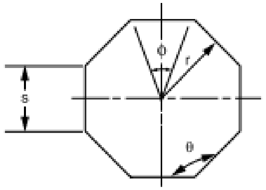
Parallelogram



$$P = 2(a+b)$$
$$d_1 = \sqrt{a^2+b^2-2ab(\cos\phi)}$$
$$d_2 = \sqrt{a^2+b^2+2ab(\cos\phi)}$$
$$d_1^2+d_2^2 = 2(a^2+b^2)$$
$$A = ah = ab(\sin\phi)$$

If $a = b$, the parallelogram is a rhombus.

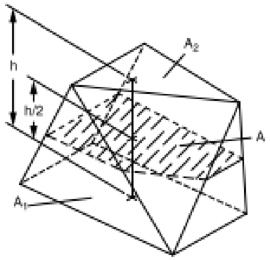
Regular Polygon (n equal sides)



$$\phi = 2\pi/n$$
$$\theta = \left[\frac{\pi(n-2)}{n}\right] = \pi\left(1-\frac{2}{n}\right)$$
$$P = ns$$
$$s = 2r[\tan(\phi/2)]$$
$$A = (nsr)/2$$

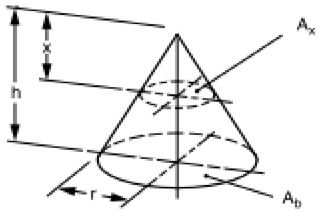
Source: Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publishing, 1967.

Prismoid



$$V = (h/6)(A_1 + A_2 + 4A)$$

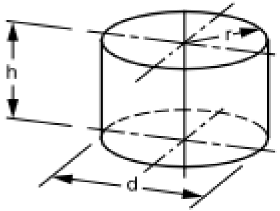
Right Circular Cone



$$V = (\pi r^2 h)/3$$
$$A = \text{side area} + \text{base area} = \pi r\left(r + \sqrt{r^2+h^2}\right)$$
$$A_x : A_b = x^2 : h^2$$

Source: Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publish

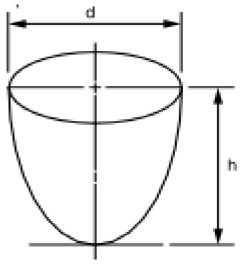
Right Circular Cylinder



$$V = \pi r^2 h = \frac{\pi d^2 h}{4}$$
$$A = \text{side area} + \text{end areas} = 2\pi r(h + r)$$

Source: Gieck, K., and R. Gieck, *Engineering Formulas*, 6th ed., Gieck Publish

Paraboloid of Revolution



$$V = \frac{\pi d^2 h}{8}$$

❖ A **conic section** is any of several kinds of curves that can be produced by passing a plane through a cone as shown in Fig, 4,14,

Conic Section Equation

The general form of the conic section equation is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

where not both A and C are zero.

If $B^2 - 4AC < 0$, an ellipse is defined.

If $B^2 - 4AC > 0$, a hyperbola is defined.

If $B^2 - 4AC = 0$, the conic is a parabola.

If $A = C$ and $B = 0$, a circle is defined.

If $A = B = C = 0$, a straight line is defined.

$$x^2 + y^2 + 2ax + 2by + c = 0$$

is the normal form of the conic section equation, if that conic section has a principal axis parallel to a coordinate axis.

$$h = -a;\ k = -b$$

$$r = \sqrt{a^2 + b^2 - c}$$

If $a^2 + b^2 - c$ is positive, a circle, center $(-a, -b)$.

If $a^2 + b^2 - c$ equals zero, a point at $(-a, -b)$.

If $a^2 + b^2 - c$ is negative, locus is imaginary.

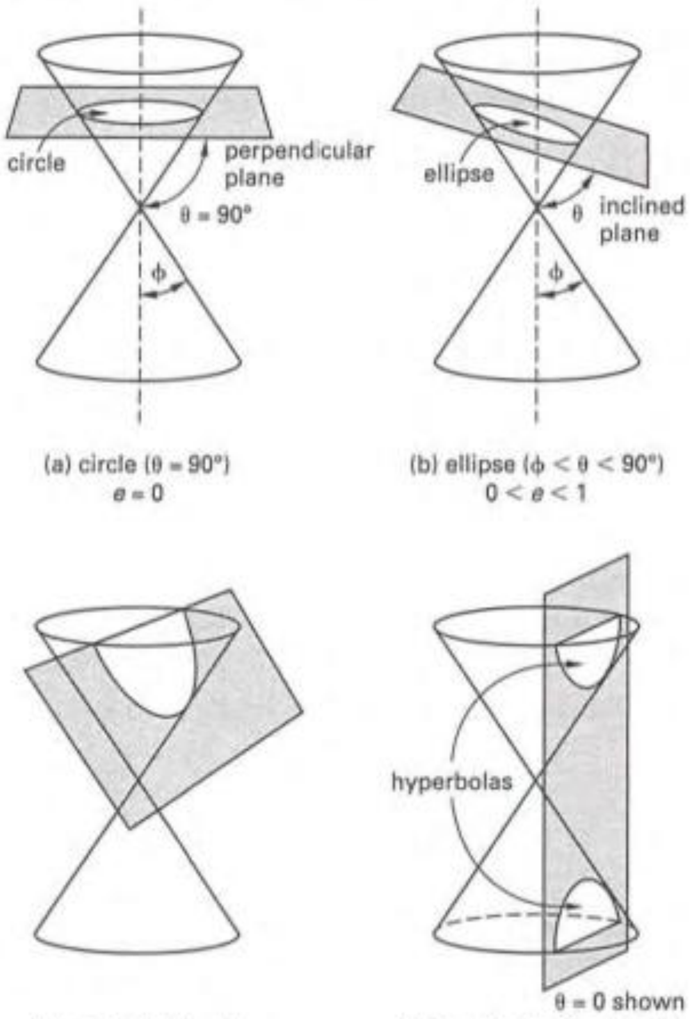
❖ **Eccentricity of a Cutting Plane**

Description

If θ is the angle between the vertical axis and the cutting plane and ϕ is the *cone-generating angle*, then the *eccentricity*, e, of the conic section is given by Eq. 4.37.

$$e = \cos\theta/(\cos\phi) \qquad 4.37$$

Figure 4.14 *Conic Sections Produced by Cutting Planes*



(a) circle (θ = 90°) $e = 0$

(b) ellipse (ϕ < θ < 90°) $0 < e < 1$

(c) parabola (θ = ϕ) $e = 1$

(d) hyperbolas (0 ≤ θ < ϕ) $e > 1$

Example

What kind of conic section is described by the following equation?

$$4x^2 - y^2 + 8x + 4y = 15$$

(A) circle

(B) ellipse

(C) parabola

(D) hyperbola

Solution

The general form of a conic section is given by Eq. 4.38 as

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

In this case, $A=4$, $B=0$, and $C=-1$. Since $A \neq C$, the conic section is not a circle or line.

Calculate the discriminant.

$$B^2 - 4AC = (0)^2 - (4)(4)(-1) = 16$$

This is greater than zero, so the section is a hyperbola.

The answer is (D).

Analytic Geometry

❖ A **conic section** is any of several kinds of curves that can be produced by passing a plane through a cone as shown in Fig, 4,14,

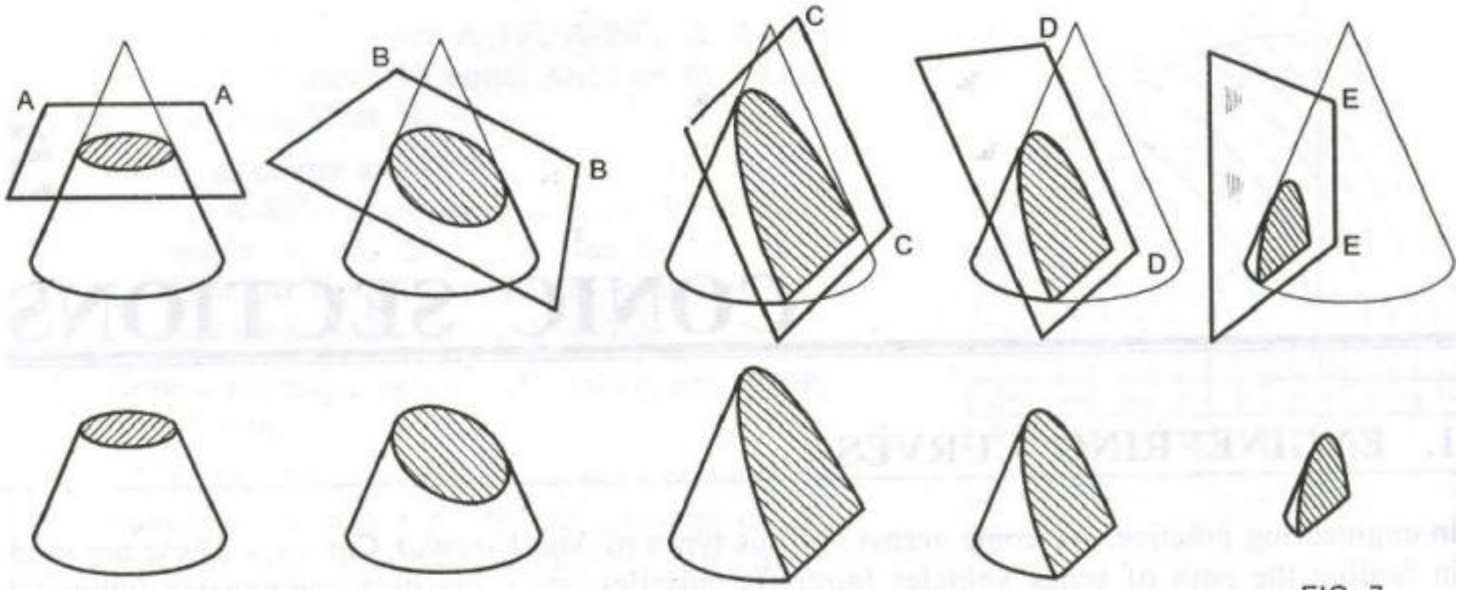


FIG. 3 CIRCLE

FIG. 4 ELLIPSE

FIG. 5 PARABOLA

FIG. 6 HYPERBOLA

FIG. 7 RECTANGULAR HYPERBOLA

Conic Section Equation

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where not both A and C are zero.

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$$x^2 + y^2 + 2ax + 2by + c = 0$$

is the normal form of the conic section equation, if that conic section has a principal axis parallel to a coordinate axis.

$$h = -a;\ k = -b$$

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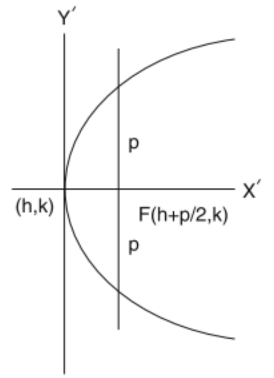
Description

If θ is the angle between the vertical axis and the cutting plane and ϕ is the *cone-generating angle*, then the *eccentricity*, e, of the conic section is given by Eq. 4.37.

$$e = \cos\theta / (\cos\phi) \qquad 4.37$$

Case 1. Parabola $e = 1$: $(y-k)^2 = 2p(x-h)$; Center at (h, k) is the standard form of the equation. When $h = k = 0$, Focus: $(p/2, 0)$; Directrix: $x = -p/2$



Problem 1.4 f) ________ represents a parabola with an opening on the positive y-axis.

(A) $(x-4)^2 = 2(y-2)$

(B) $(x-4)^2 = -2(y-2)$

(C) $(y-2)^2 = -4(x-4)$

(D) $(y-2)^2 = 4(x-4)$

1.4 f) CORRECT ANSWER - A

Standard forms of parabola equations are given below.

$(y-k)^2 = 2p(x-h)$ Parabola opening on the positive x-axis.

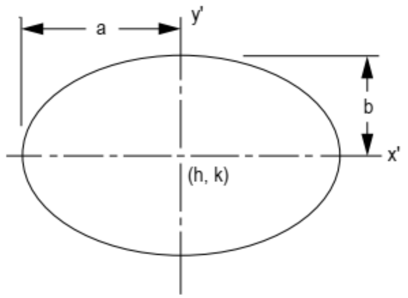
$(y-k)^2 = -2p(x-h)$ Parabola opening on the negative x-axis.

$(x-h)^2 = 2p(y-k)$ Parabola opening on the positive y-axis.

$(x-h)^2 = -2p(y-k)$ Parabola opening on the negative y-axis.

It can be observed that $(x-4)^2 = 2(y-2)$ represents a parabola opening on the positive y-axis.

Case 2. Ellipse $e < 1$:



$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$; Center at (h, k) is the standard form of the equation. When $h = k = 0$,

Eccentricity: $e = \sqrt{1-(b^2/a^2)} = c/a$

$b = a\sqrt{1-e^2}$;

Focus: $(\pm ae, 0)$; Directrix: $x = \pm a/e$

Problem 1.4 g) __________ represents an ellipse with a vertical major axis.

(A) $\frac{(x-2)^2}{144} + \frac{(y-4)^2}{100} = 1$

(B) $\frac{(x-2)^2}{100} + \frac{(y-4)^2}{144} = 1$

(C) $\frac{(x-2)^2}{144} - \frac{(y-4)^2}{100} = 1$

(D) $\frac{(y-4)^2}{100} - \frac{(x-2)^2}{144} = 1$

1.4 g) CORRECT ANSWER - B

Standard forms of ellipse equations are given below.

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ $\quad a > b$ Ellipse with a horizontal major axis (wide ellipse).

$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ $\quad a < b$ Ellipse with a vertical major axis (tall ellipse).

It can be observed that $\frac{(x-2)^2}{100} + \frac{(y-4)^2}{144} = 1$ represents a tall ellipse with a vertical major axis.

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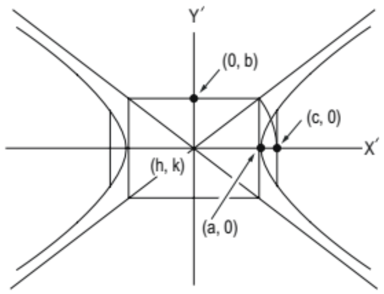
Case 3. Hyperbola $e > 1$:



$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1;$$

Center at (h, k) is the standard form of the equation. When $h = k = 0$,

Eccentricity: $e = \sqrt{1 + (b^2/a^2)} = c/a$

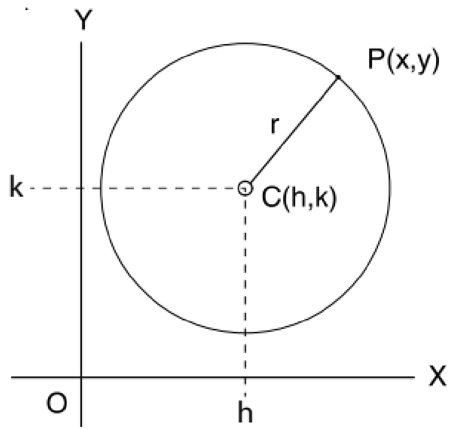
$$b = a\sqrt{e^2 - 1};$$

Focus: $(\pm ae, 0)$; Directrix: $x = \pm a/e$

Case 4. Circle $e = 0$:

$(x-h)^2 + (y-k)^2 = r^2$; Center at (h, k) is the standard form of the equation with radius

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$



Problem 1.4 h) ___________ represents a hyperbola with upward/downward openings.

(A) $\frac{(x-2)^2}{144} + \frac{(y-4)^2}{100} = 1$ (B) $\frac{(x-2)^2}{100} + \frac{(y-4)^2}{144} = 1$

(C) $\frac{(x-2)^2}{144} - \frac{(y-4)^2}{100} = 1$ (D) $\frac{(y-4)^2}{100} - \frac{(x-2)^2}{144} = 1$

Standard forms of hyperbola equations are given below.

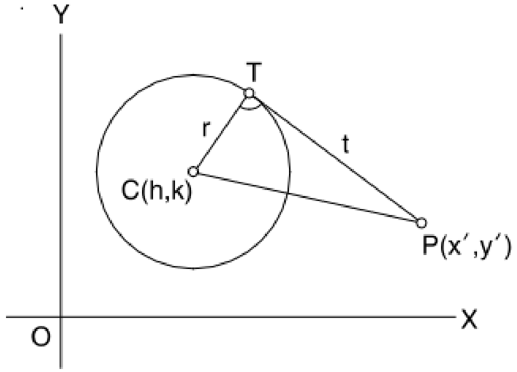
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ Hyperbola opening on the x-axis (sideways).

$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ Hyperbola opening on the y-axis (upwards/downwards).

It can be observed that $\frac{(y-4)^2}{100} - \frac{(x-2)^2}{144} = 1$ represents a hyperbola that opens upwards/downwards.

Length of the tangent line from a point on a circle to a point (x', y'):

$$t^2 = (x' - h)^2 + (y' - k)^2 - r^2$$



❖ STRAIGHT LINES

The general form of the equation is

$$Ax + By + C = 0$$

The standard form of the equation is

$$y = mx + b,$$

which is also known as the *slope-intercept* form.

The *point-slope* form is

$$y - y_1 = m(x - x_1)$$

Given two points: slope,

$$m = (y_2 - y_1)/(x_2 - x_1)$$

The angle between lines with slopes m_1 and m_2 is

$$\alpha = \arctan\left[(m_2 - m_1)/(1 + m_2 \cdot m_1)\right]$$

Two lines are perpendicular if $m_1 = -1/m_2$

The distance between two points is

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Example

Find the slope of the line that passes through points (−3, 2) and (5, −2).

(A) −2

(B) −0.5

(C) 0.5

(D) 2

Solution

Use Eq. 4.30.

$$m = (y_2 - y_1)/(x_2 - x_1) = \frac{-2 - 2}{5 - (-3)} = -0.5$$

The answer is (B).

Example

A line goes through the point (4, −6) and is perpendicular to the line $y = 4x + 10$. What is the equation of the line?

(A) $y = -\frac{1}{4}x - 20$

(B) $y = -\frac{1}{4}x - 5$

(C) $y = \frac{1}{5}x + 5$

(D) $y = \frac{1}{4}x + 5$

Solution

The slopes of two lines that are perpendicular are related by

$$m_1 = -1/m_2$$

The slope of the line perpendicular to the line with slope $m_1 = 4$ is

$$m_2 = -1/m_1 = -\frac{1}{4}$$

The equation of the line is in the form $y = mx + b$. $m = -1/4$, and a known point is $(x, y) = (4, -6)$.

$$-6 = \left(-\tfrac{1}{4}\right)(4) + b$$
$$b = -6 - \left(-\tfrac{1}{4}\right)(4)$$
$$= -5$$

The equation of the line is

$$y = -\tfrac{1}{4}x - 5$$

The answer is (B).

Analytic Geometry

❖ **STRAIGHT LINES**

The general form of the equation is

$$Ax + By + C = 0$$

The standard form of the equation is

$$y = mx + b,$$

which is also known as the *slope-intercept* form.

The *point-slope* form is

$$y - y_1 = m(x - x_1)$$

Given two points: slope,

$$m = (y_2 - y_1)/(x_2 - x_1)$$

The angle between lines with slopes m_1 and m_2 is

$$\alpha = \arctan\left[(m_2 - m_1)/(1 + m_2 \cdot m_1)\right]$$

Two lines are perpendicular if $m_1 = -1/m_2$

The distance between two points is

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Example

What is the slope of the line defined by $y - x = 5$?

(A) −1
(B) −1/5
(C) 1/4
(D) 1

Solution

The standard (or slope-intercept) form of the equation of a straight line is $y = mx + b$, where m is the slope and b is the y-intercept. Rearrange the given equation into standard form.

$$y - x = 5$$

$$y = x + 5$$

The slope, m, is the coefficient of x, which is 1.

The answer is (D).

Example

The angle between the line $y = -7x + 12$ and the line $y = 3x$ is most nearly

(A) 22°
(B) 27°
(C) 33°
(D) 37°

Solution

Use Eq. 4.35.

$$\alpha = \arctan[(m_2 - m_1)/(1 + m_2 \cdot m_1)]$$

$$= \arctan \frac{-7 - 3}{1 + (-7)(3)} = \arctan 0.5$$

$$= 26.57° \quad (27°)$$

The answer is (B).

❖ **STRAIGHT LINES**

The general form of the equation is

$$Ax + By + C = 0$$

The standard form of the equation is

$$y = mx + b,$$

which is also known as the *slope-intercept* form.

The *point-slope* form is

$$y - y_1 = m(x - x_1)$$

Given two points: slope,

$$m = (y_2 - y_1)/(x_2 - x_1)$$

The angle between lines with slopes m_1 and m_2 is

$$\alpha = \arctan\,[(m_2 - m_1)/(1 + m_2 \cdot m_1)]$$

Two lines are perpendicular if $m_1 = -1/m_2$

The distance between two points is

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Example

What is the distance between point P at $(1, -3, 5)$ and point Q at $(-3, 4, -2)$?

(A) $\sqrt{10}$
(B) $\sqrt{14}$
(C) 8
(D) $\sqrt{114}$

Solution

The distance between points P and Q is

$$\begin{aligned} d_{PQ} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(-3 - 1)^2 + \left(4 - (-3)\right)^2 + (-2 - 5)^2} \\ &= \sqrt{114} \end{aligned}$$

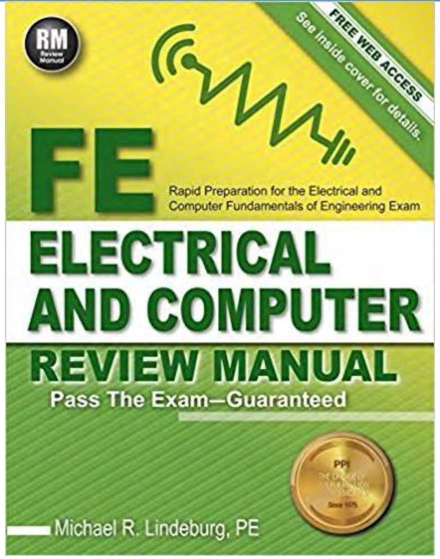
The answer is (D).

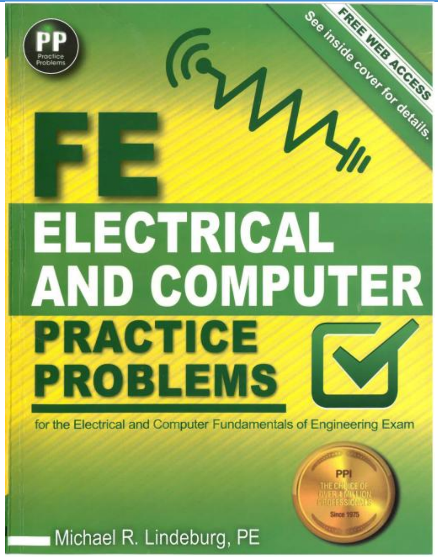
Thank you

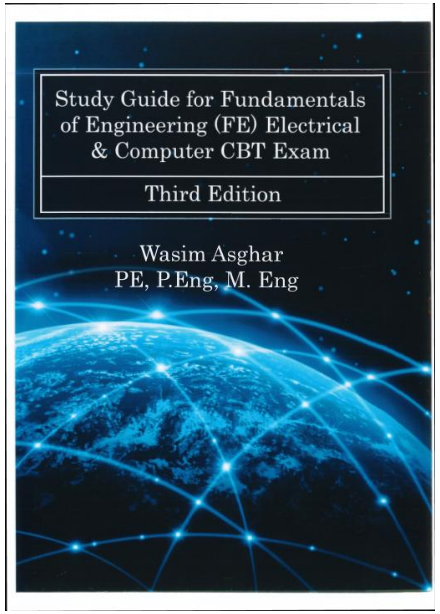
References

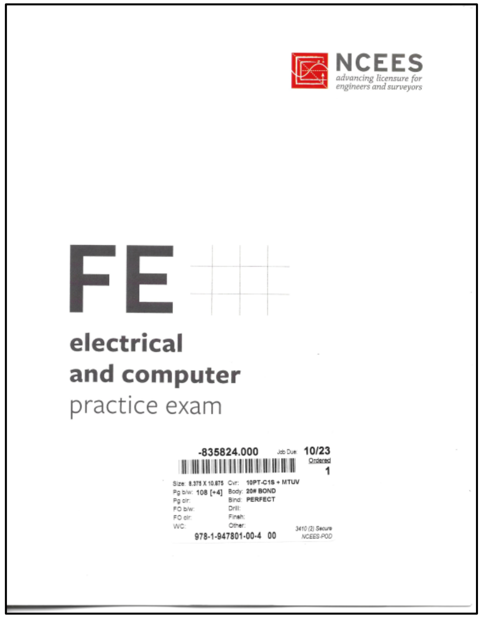
References

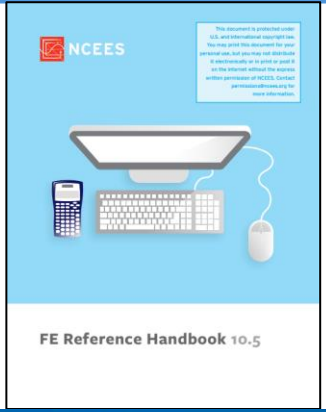
المرجع الخاص بالمجلس الوطني الأمريكي وبيكون معاك أثناء الاختبار

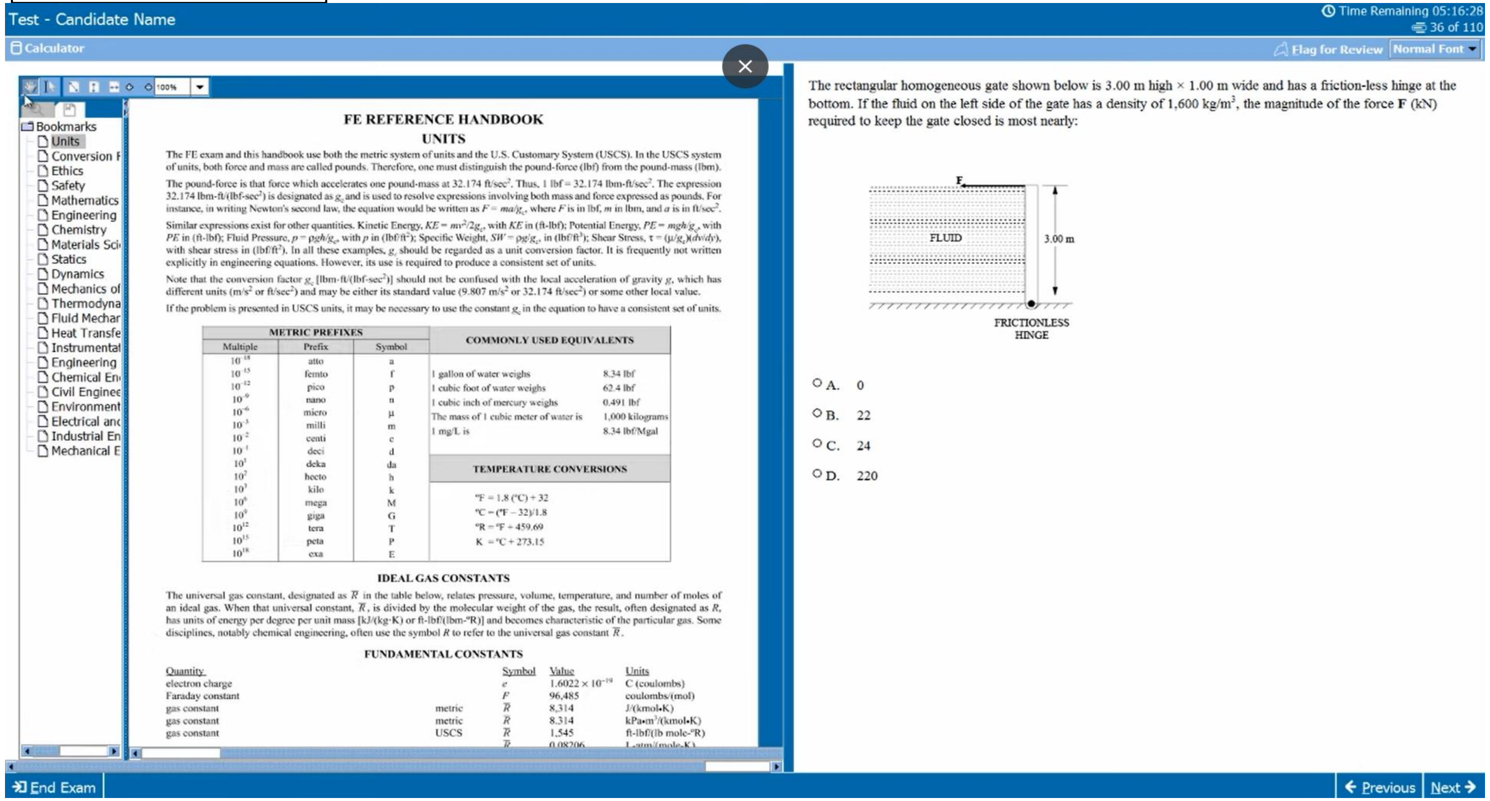












Practice problems (Mathematics)

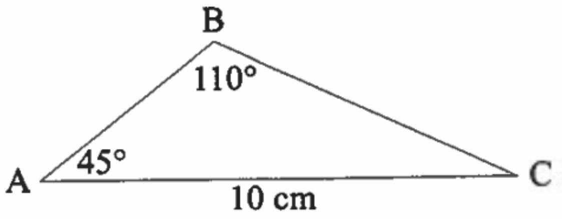
Practice problems (Mathematics)

1. Mathematics **11–17**

A. Algebra and trigonometry
B. Complex numbers
C. Discrete mathematics
D. Analytic geometry
E. Calculus (e.g., differential, integral, single-variable, multivariable)
F. Ordinary differential equations
G. Linear algebra
H. Vector analysis

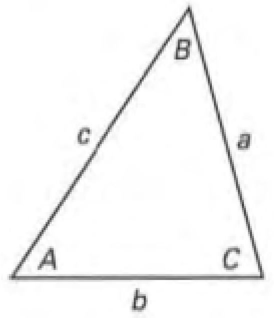
Practice problems (Mathematics)

1. In the following triangle, the length (cm) of Side AB is most nearly:



- ○ A. 4.5
- ○ B. 7.1
- ○ C. 7.5
- ○ D. 192

Figure 5.7 *General Triangle*



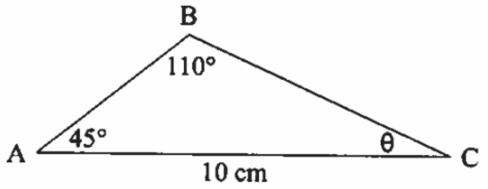
1. Refer to the Law of Sines equation in the Mathematics chapter of the *FE Reference Handbook*.

$\theta = 180 - 110 - 45 = 25°$

Law of Sines:

$$\frac{10}{\sin 110°} = \frac{AB}{\sin 25°}$$

AB = 4.497 cm



THE CORRECT ANSWER IS: A

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Practice problems (Mathematics)

Problem 1.1 a) Solve the following logarithmic equation for x:

$$\log_3(12x - 12) - \log_3(x) = 2$$

(A) 2 (B) 4

(C) 3 (D) 0

1.1 a) CORRECT ANSWER – B

$\log_3(12x - 12) - \log_3(x) = 2$

According to the logarithmic identities given in NCEES® FE Reference Handbook:

$$\log x - \log y = \log\frac{x}{y} \rightarrow \log_3(12x - 12) - \log_3(x) = \log_3\frac{(12x - 12)}{(x)}$$

The given logarithmic function can be rearranged as follows:

$$\log_3\frac{(12x - 12)}{(x)} = 2$$

$$\frac{(12x - 12)}{(x)} = 3^2 \rightarrow 12x - 12 = 9x \rightarrow 3x = 12 \rightarrow x = 4$$

$\log_b(x) = c \quad [b^c = x]$

$\ln x \quad [\text{base} = e]$

$\log x \quad [\text{base} = 10]$

Inverse Property

$b^{\log_b x} = x$ and $\log_b b^x = x$

$4^{\log_4 6} = 6$

$\log_4 4^6 = 6$

$\log_b b^n = n$

$\log x^c = c \log x$

$x^c = \text{antilog}(c \log x)$

$\log xy = \log x + \log y$

$\log_b b = 1$

$\log 1 = 0$

$\log x/y = \log x - \log y$

$\log_b x = (\log_a x)/(\log_a b)$

Practice problems (Mathematics)

Problem 1.1 b) _______ logarithm(s) do not exist and cannot be calculated.

(A) $\log(0.001)$

(B) $\log(0)$

(C) $\log(-10)$

(D) $\log(10)$

1.1 b) CORRECT ANSWERS – B, C

Logarithm can only be calculated for positive real numbers.

Logarithms of 0 and negative real numbers are undefined.

Practice problems (Mathematics)

Problem 1.1 c) Solve the following logarithmic equation for x:

$$\log_3(x+1) + \log_3(x-1) = 1$$

(A) 0

(B) 2

(C) -2

(D) 12

1.1 c) CORRECT ANSWER – B

$\log_3(x+1) + \log_3(x-1) = 1$

According to the logarithmic identities given in NCEES® FE Reference Handbook:

$\log x + \log y = \log xy \rightarrow \log_3(x+1) + \log_3(x-1) = \log_3[(x+1)(x-1)] = \log_3(x^2-1)$

The given logarithmic function can be rearranged as follows:

$\log_3(x^2-1) = 1$

Taking anti-log on both sides of the equation results in following:

$x^2 - 1 = 3^1 = 3$

$x^2 = 4 \rightarrow x = 2, -2$

It is important to validate these results by substitution to see if values of x are acceptable solutions.

$\log_3(2+1) + \log_3(2-1) = 1 + 0 = 1$ Therefore, $x = 2$ is a valid solution.

$\log_3(-2+1) + \log_3(-2-1) = \log_3(-1) + \log_3(-3)$ $x = 2$ is an invalid solution because logarithm of negative real number is undefined.

$\log_b(x) = c \quad [b^c = x]$

$\ln x \quad [\text{base} = e]$

$\log x \quad [\text{base} = 10]$

Inverse Property

$b^{\log_b x} = x$ and $\log_b b^x = x$

$4^{\log_4 6} = 6$

$\log_4 4^6 = 6$

$\log_b b^n = n$

$\log x^c = c \log x$

$x^c = \text{antilog}(c \log x)$

$\log xy = \log x + \log y$

$\log_b b = 1$

$\log 1 = 0$

$\log x/y = \log x - \log y$

$\log_b x = (\log_a x)/(\log_a b)$

2. The term $\frac{(1-i)^2}{(1+i)^2}$, where $i=\sqrt{-1}$ is most nearly:

- ○ A. $1+i$
- ○ B. 0
- ○ C. $-1+i$
- ○ D. -1

2. Refer to the Algebra of Complex Numbers section in the Mathematics chapter of the *FE Reference Handbook.*

$$\frac{(1-i)^2}{(1+i)^2}=\frac{1-2i+i^2}{1+2i+i^2}=\frac{1-1-2i}{1-1+2i}=\frac{-i}{i}=-1$$

THE CORRECT ANSWER IS: D

Practice problems (Mathematics)

Problem 1.2 e) $3 + 4j$ can be expressed as ___________.

(A) $5(\cos 53° + j \sin 53°)$

(B) $5e^{j53}$

(C) $5\underline{/53°}$

(D) All of the above

$z = x + jy$ Rectangular form

$z = r\underline{/\phi}$ Polar form

$z = re^{j\phi}$ Exponential form

$$r = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\phi, \qquad y = r\sin\phi$$

1.2 e) CORRECT ANSWER - D

This problem falls under the category of 'Algebra of Complex Numbers'.

$5(\cos 53^0 + j \sin 53^0)$ is rectangular form of $3 + 4j$.

$5e^{j53}$ is Euler's form of $3 + 4j$.

$5\underline{/53^0}$ is polar form of $3 + 4j$.

Therefore, all options are accurate representations of $3 + 4j$.

Practice problems (Mathematics)

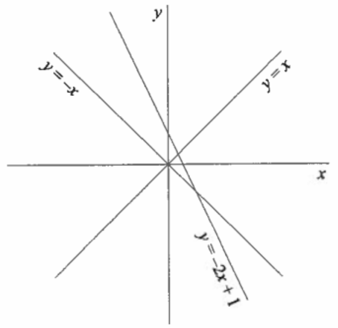
4. Three lines are defined by the three equations:

$$x + y = 0$$
$$x - y = 0$$
$$2x + y = 1$$

The three lines form a triangle with vertices at:

- ○ A. $(0, 0), \left(\frac{1}{3}, \frac{1}{3}\right), (1, -1)$
- ○ B. $(0, 0), \left(\frac{2}{3}, \frac{2}{3}\right), (-1, -1)$
- ○ C. $(1, 1), (1, -1), (2, 1)$
- ○ D. $(1, 1), (3, -3), (-2, -1)$

4. Refer to the Straight Line section in the Mathematics chapter of the *FE Reference Handbook.*



$y = -x$
$y = x$
$y = -2x + 1$

From graph one, the intersection is at (0, 0), so Options C and D are incorrect.

Also, the second intersection is at (1, –1), so the vertices are at $(0, 0), \left(\frac{1}{3}, \frac{1}{3}\right), (1, -1)$.

THE CORRECT ANSWER IS: A

The general form of the equation is

$$Ax + By + C = 0$$

The standard form of the equation is

$$y = mx + b,$$

which is also known as the *slope-intercept* form.

The *point-slope* form is

$$y - y_1 = m(x - x_1)$$

Given two points: slope,

$$m = (y_2 - y_1)/(x_2 - x_1)$$

The angle between lines with slopes m_1 and m_2 is

$$\alpha = \arctan\left[(m_2 - m_1)/(1 + m_2 \cdot m_1)\right]$$

Two lines are perpendicular if $m_1 = -1/m_2$

The distance between two points is

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Practice problems (Mathematics)

5. The only point of inflection on the curve representing the equation $y = x^3 + x^2 - 3$ is at:

- A. $x = -\frac{2}{3}$
- B. $x = -\frac{1}{3}$
- C. $x = 0$
- D. $x = \frac{1}{3}$

5. Refer to the Test for a Point of Inflection section in the Mathematics chapter of the *FE Reference Handbook.*

$f(x) = x^3 + x^2 - 3$

$f'(x) = 3x^2 + 2x$

$f''(x) = 6x + 2$
$6x + 2 = 0$
$x = -1/3$

$f''(x)$ negative below $x = -1/3$
$f''(x)$ positive above $x = -1/3$

Since $f''(x) = 0$ and $f''(x)$ changes sign at $x = -1/3$, the inflection point is at $x = -1/3$.

THE CORRECT ANSWER IS: B

Test for a Minimum

$f'(a) = 0$
$f''(a) > 0$

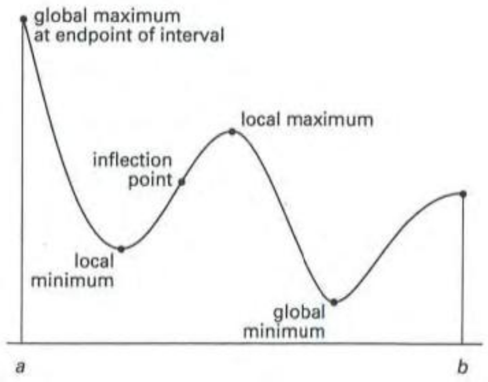
Test for a Maximum

$f'(a) = 0$
$f''(a) < 0$

Test for a Point of Inflection

$f''(a) = 0$

Figure 7.1 *Critical Points*



Practice problems (Mathematics)

6. Given the function $f(x, y) = x^2 + xy + y^2$, solve for $\frac{\partial f}{\partial y}$.

- A. $2x + x + y + 2y$
- B. $2x + y$
- C. $2y$
- D. $x + 2y$

$$z = f(x, y)$$

$$\frac{\partial z}{\partial x} = \frac{\partial f(x, y)}{\partial x}$$

6. Refer to the Partial Derivative section in the Mathematics chapter of the *FE Reference Handbook*.

$$\frac{\partial f}{\partial y} = \frac{\partial (x^2)}{\partial y} + \frac{\partial (xy)}{\partial y} + \frac{\partial (y^2)}{\partial y}$$

$$= 0 + x + 2y$$

$$= x + 2y$$

THE CORRECT ANSWER IS: D

Practice problems (Mathematics)

7. The following equation describes a second-order system:

$$\frac{d^2y}{dt^2}+6\frac{dy}{dt}+25y=x(t)$$

The system may be described as:

- ○ A. nonlinear
- ○ B. overdamped
- ○ C. critically damped
- ○ D. underdamped

7. $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+25y=x(t)$

The characteristic equation is $D^2+6D+25=0$

Referring to the Second-Order Linear Homogeneous Differential Equations with Constant Coefficients section in the Mathematics chapter of the *FE Reference Handbook*:

$a = 6$
$a^2 = 36$
$b = 25$
$4b = 100$

Since $a^2 = 36$ is less than $4b = 100$, the system is underdamped.

THE CORRECT ANSWER IS: D

$$y' + ay = 0$$
$$y = Ce^{-at}$$

$$r_{1,2} = \frac{-a \pm \sqrt{a^2-4b}}{2}$$
$$y = C_1e^{r_1x} + C_2e^{r_2x}$$
$$y = (C_1 + C_2x)e^{r_1x}$$
$$y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x)$$
$$\alpha = -a/2$$
$$\beta = \frac{\sqrt{4b-a^2}}{2}$$

$$y'' + ay' + by = 0$$
$$(r^2 + ar + b)Ce^{rx} = 0$$
$$r^2 + ar + b = 0$$

If $a^2 > 4b$, the solution is of the form (overdamped)

$$y = C_1e^{r_1x} + C_2e^{r_2x}$$

If $a^2 = 4b$, the solution is of the form (critically damped)

$$y = (C_1 + C_2x)e^{r_1x}$$

If $a^2 < 4b$, the solution is of the form (underdamped)

$$y = e^{\alpha x}(C_1\cos\beta x + C_2\sin\beta x), \text{ where}$$
$$\alpha = -a/2$$
$$\beta = \frac{\sqrt{4b-a^2}}{2}$$

Practice problems (Mathematics)

Problem 1.6 a) Solve the following 1st order linear differential equation with given initial values.

$$2y' + 4y = 0 \qquad y(0) = 6$$

(A) $y = 3e^{-4t}$ (B) $y = 6e^{-4t}$

(C) $y = 3e^{-2t}$ (D) $y = 6e^{-2t}$

1.6 a) CORRECT ANSWER - D

This problem falls under the category of 'Differential Equations'.

$$2y' + 4y = 0 \qquad y(0) = 6$$

Standard form of 1st order differential equation with constant coefficient is given below:

$$y' + ay = 0$$

Solution of standard equation is $y = Ce^{-at}$.

Divide given equation by 2 to convert it into standard form $y' + 2y = 0$

In our case, $a = 2$. Therefore, general solution is $y = Ce^{-2t}$.

We can calculate coefficient using initial conditions as shown below:

$$y(0) = 6 \qquad 6 = Ce^{(-2)(0)} \rightarrow C = 6$$

Therefore, $y = 6e^{-2t}$

If $a^2 > 4b$, the solution is of the form (overdamped)

$$y = C_1e^{r_1x} + C_2e^{r_2x}$$

If $a^2 = 4b$, the solution is of the form (critically damped)

$$y = (C_1 + C_2x)e^{r_1x}$$

If $a^2 < 4b$, the solution is of the form (underdamped)

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x), \text{ where}$$

$$\alpha = -a/2$$

$$\beta = \frac{\sqrt{4b - a^2}}{2}$$

$$y' + ay = 0$$

$$y = Ce^{-at}$$

$$r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$$y = C_1e^{r_1x} + C_2e^{r_2x}$$

$$y = (C_1 + C_2x)e^{r_1x}$$

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

$$\alpha = -a/2$$

$$\beta = \frac{\sqrt{4b - a^2}}{2}$$

$$y'' + ay' + by = 0$$

$$(r^2 + ar + b)Ce^{rx} = 0$$

$$r^2 + ar + b = 0$$

Practice problems (Mathematics)

8. The general solution to $y'' + 4y' + 4y = 0$ is:

- A. C_1e^{-4x}
- B. C_1e^{-2x}
- C. $e^{-4x}(C_1 + C_2x)$
- D. $e^{-2x}(C_1 + C_2x)$

8. Refer to the Differential Equations section in the Mathematics chapter of the *FE Reference Handbook*. The characteristic equation for a second-order linear homogeneous differential equation is:

$$r^2 + ar + b = 0$$

In this problem, $a = 4$ and $b = 4$

$$r^2 + 4r + 4 = 0$$

In solving the characteristic equation, it is noted that there are repeated real roots: $r_1 = r_2 = -2$

Because $a^2 = 4b$, the solution for this critically damped system is:

$$y(x) = (C_1 + C_2x)\,e^{-2x}$$

THE CORRECT ANSWER IS: D

If $a^2 > 4b$, the solution is of the form (overdamped)

$$y = C_1e^{r_1x} + C_2e^{r_2x}$$

If $a^2 = 4b$, the solution is of the form (critically damped)

$$y = (C_1 + C_2x)e^{r_1x}$$

If $a^2 < 4b$, the solution is of the form (underdamped)

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x), \text{ where}$$

$$\alpha = -a/2$$

$$\beta = \frac{\sqrt{4b - a^2}}{2}$$

$$y' + ay = 0$$

$$y = Ce^{-at}$$

$$y'' + ay' + by = 0$$

$$(r^2 + ar + b)Ce^{rx} = 0$$

$$r^2 + ar + b = 0$$

$$r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$$y = C_1e^{r_1x} + C_2e^{r_2x}$$

$$y = (C_1 + C_2x)e^{r_1x}$$

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

$$\alpha = -a/2$$

$$\beta = \frac{\sqrt{4b - a^2}}{2}$$

Practice problems (Mathematics)

Problem 1.6 e) Solve the following 2nd order linear differential equation with initial conditions.

$$y'' + 2y' - 8y = 0 \qquad y(0) = 6, \qquad y'(0) = 0$$

1.6d) CORRECT ANSWER - $\boldsymbol{y = 4e^{2x} + 2e^{-4x}}$

This problem falls under the category of 'Differential Equations'.

$y'' + 2y' - 8y = 0 \qquad y(0) = 6 \quad y'(0) = 0$

We can use method of undetermined coefficients, as explained in NCEES® FE Reference Hand

The solution will be of form $y = Ce^{rx}$ with characteristic equation: $r^2 + 2r - 8 = 0$

Comparing this equation with standard form $r^2 + ar + b = 0$ shows that $a = 2, b = -8$.

Solving $r^2 + 2r - 8 = 0$ results in $r_1 = 2, r_2 = -4$

Since $a^2 = 4 > 4b = -32$, solution is overdamped and can be represented as shown below:

$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \quad y' = C_1 r_1 e^{r_1 x} + C_2 r_2 e^{r_2 x}$

We can calculate coefficients using initial conditions as shown below:

$y(0) = 6 \rightarrow \qquad 6 = C_1 e^{(2)(0)} + C_2 e^{(-4)(0)} = C_1 + C_2 \rightarrow C_1 = 6 - C_2$

$y'(0) = 0 \rightarrow \qquad 0 = C_1 2e^{(2)(0)} + C_2(-4)e^{(-4)(0)} = 2C_1 - 4C_2 = 0$

$2C_1 - 4C_2 = 0 \rightarrow 2(6 - C_2) - 4C_2 = 0$

$12 - 2C_2 - 4C_2 = 0$

Solving these equations results in $C_1 = 4$ and $C_2 = 2$. Therefore, $y = 4e^{2x} + 2e^{-4x}$

If $a^2 > 4b$, the solution is of the form (overdamped)

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

If $a^2 = 4b$, the solution is of the form (critically damped)

$$y = (C_1 + C_2 x)e^{r_1 x}$$

If $a^2 < 4b$, the solution is of the form (underdamped)

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x), \text{ where}$$

$$\alpha = -a/2$$

$$\beta = \frac{\sqrt{4b - a^2}}{2}$$

$$y' + ay = 0$$

$$y = Ce^{-at}$$

$$r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

$$y = (C_1 + C_2 x)e^{r_1 x}$$

$$y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

$$\alpha = -a/2$$

$$\beta = \frac{\sqrt{4b - a^2}}{2}$$

$$y'' + ay' + by = 0$$

$$(r^2 + ar + b)Ce^{rx} = 0$$

$$r^2 + ar + b = 0$$

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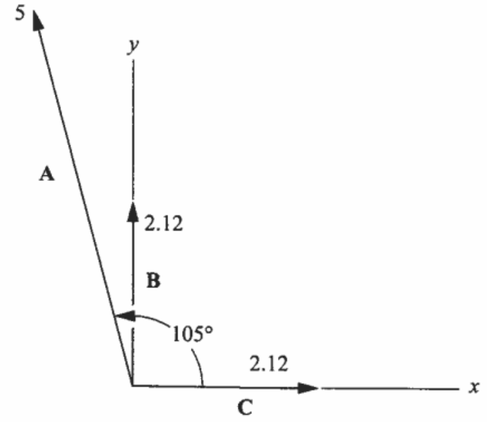
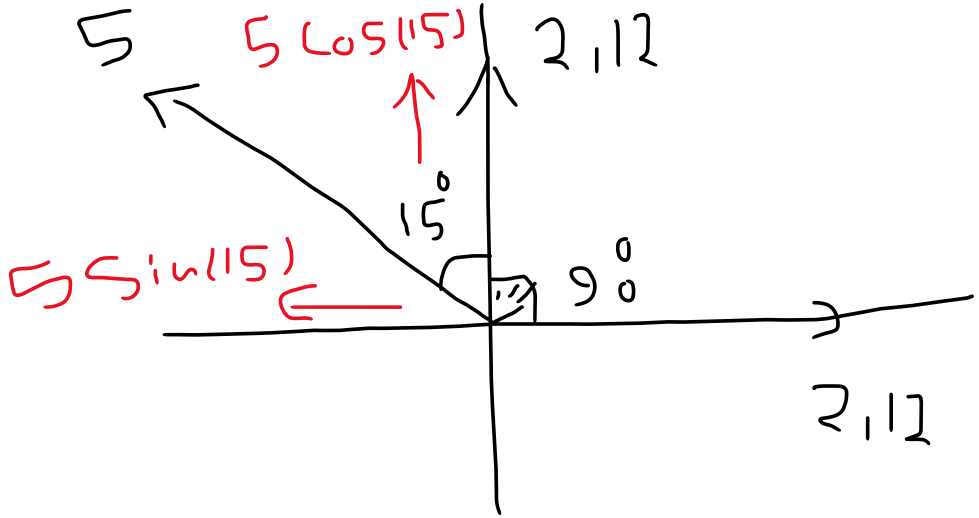
Practice problems (Mathematics)

9. The magnitude of the resultant of the three coplanar vectors **A**, **B**, and **C**, is most nearly:



- ○ A. 7.0
- ○ B. 7.8
- ○ C. 9.2
- ○ D. 10.3



$R_x = 2{,}12 - 5\sin(15) = .83$

$R_y = 2{,}12 + 5\cos(15) = 6{,}95$

9. Refer to the Resolution of a Force section in the Statics chapter of the *FE Reference Handbook*.

$R_x = \sum F_{xi}, \quad R_y = \sum F_{yi}, \quad i = 1,2,3$

$R_x = 2.12 + 5\cos 105° = 2.12 - 1.29 = 0.83$

$R_y = 2.12 + 5\sin 105° = 2.12 + 4.83 = 6.95$

$R = \sqrt{R_x^2 + R_y^2} = \sqrt{0.83^2 + 6.95^2} = 6.999$

THE CORRECT ANSWER IS: A

Practice problems (Mathematics)

10. Which of the following is a unit vector perpendicular to the plane determined by the vectors $\mathbf{A} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{B} = \mathbf{i} + \mathbf{j} - \mathbf{k}$?

- A. $-2\mathbf{i} + \mathbf{j} - \mathbf{k}$
- B. $\frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$
- C. $\frac{1}{\sqrt{6}}(-2\mathbf{i} + \mathbf{j} - \mathbf{k})$
- D. $\frac{1}{\sqrt{6}}(-2\mathbf{i} - \mathbf{j} - \mathbf{k})$

10. Refer to the Vectors section in the Mathematics chapter of the *FE Reference Handbook*.

The cross product of Vectors **A** and **B** is a vector perpendicular to **A** and **B**.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 0 \\ 1 & 1 & -1 \end{vmatrix} = \mathbf{i}(-4) - \mathbf{j}(-2-0) + \mathbf{k}(2-4) = -4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$$

To obtain a unit vector, divide by the magnitude.

$$\text{Magnitude} = \sqrt{(-4)^2 + 2^2 + (-2)^2} = \sqrt{24} = 2\sqrt{6}$$

$$\frac{-4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}}{2\sqrt{6}} = \frac{-2\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{6}}$$

THE CORRECT ANSWER IS: C

Problem 1.7 i) The dot product of vectors $\vec{A}$ and $\vec{B}$ is ________

$$\vec{A} = 2i + j + 3k$$

$$\vec{B} = i + 2j + 4k$$

1.7 i) CORRECT ANSWER: 16

$\vec{A} = 2i + j + 3k \qquad \vec{B} = i + 2j + 4k$

$\vec{A}.\vec{B} = (2i + j + 3k).(i + 2j + 4k)$

$\vec{A}.\vec{B} = 2.1 + 1.2 + 3.4 = 16$

Problem 1.7 j) The cross product of vectors $\vec{A}$ and $\vec{B}$ is ________.

$$\vec{A} = 3i + 2j + k$$

$$\vec{B} = i + 4j + 0k$$

1.7 j) CORRECT ANSWER: $-4i + j + 10k$

$\vec{A} = 3i + 2j + k \qquad \vec{B} = i + 4j + 0k$

$$\vec{A} \times \vec{B} = (3i + 2j + k) \times (i + 4j + 0k) = \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ 1 & 4 & 0 \end{vmatrix}$$

$$\vec{A} \times \vec{B} = (2 \times 0 - 1 \times 4)i - (3 \times 0 - 1 \times 1)j + (3 \times 4 - 2 \times 1)k$$

$$\vec{A} \times \vec{B} = -4i + j + 10k$$

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Practice problems (Mathematics)

11. Consider the following 2 × 2 matrix A:

$$A = \begin{bmatrix} 2 & -3 \\ 4 & K \end{bmatrix}$$

The value of K for which A has no inverse is most nearly:

- A. –6
- B. –8/3
- C. –3/2
- D. 6

11. Refer to the Matrices section in the Mathematics chapter of the *FE Reference Handbook.*

$$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

A has no inverse when $|A| = 0$.

$$|A| = (2)(K) - (-3)(4) = 0; K = -6$$

THE CORRECT ANSWER IS: A

Problem 1.7 c) Calculate the product $\boldsymbol{A} \times \boldsymbol{B}$ of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ given below.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ 6 & 3 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

(A) $\begin{bmatrix} 4 & 1 \\ 8 & 2 \\ 12 & 3 \end{bmatrix}$

(B) $\begin{bmatrix} 2 & 2 \\ 4 & 4 \\ 6 & 6 \end{bmatrix}$

(C) $\begin{bmatrix} 2 & 0 \\ 8 & 2 \end{bmatrix}$

(D) **A** and **B** cannot be multiplied

1.7 c) CORRECT ANSWER - A

This problem falls under the category of 'Matrices'.

According to the problem statement:

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ 6 & 3 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{bmatrix} 2 \times 1 + 1 \times 2 & 2 \times 0 + 1 \times 1 \\ 4 \times 1 + 2 \times 2 & 4 \times 0 + 2 \times 1 \\ 6 \times 1 + 3 \times 2 & 6 \times 0 + 3 \times 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 8 & 2 \\ 12 & 3 \end{bmatrix}$$

Practice problems (Mathematics)

Problem 1.5 k) Evaluate the following indefinite integral:

$$\int \frac{4}{x+3}dx$$

(A) $\frac{1}{\sqrt{4}}\tan^{-1}\left(x\sqrt{\frac{1}{3}}\right)+C$

(B) $2\ln|2x+6|+C$

(C) $4\sqrt{x+3}+C$

(D) $4\ln|x+3|+C$

1.5 k) CORRECT ANSWER - D

This problem falls under the category of 'Integral Calculus'.

$$\int \frac{4}{x+3}dx = 4\int \frac{1}{x+3}dx$$

According to the indefinite integrals given in NCEES® FE Reference Handbook:

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|+C$$

Substituting this formula in given indefinite integral results in:

$$4\frac{1}{1}\ln|x+3| = 4\ln|x+3|+C$$

Helpful tip – Review tables of derivatives and indefinite integrals given in NCEES® FE Reference Handbook.

Problem 1.5 l) Evaluate the following indefinite integral:

$$\int (\sin^2 x + \cos^2 x)dx$$

(A) $1+C$

(B) $x/2 - \sin 2x/4 + C$

(C) $x+C$

(D) $x/2 + \sin 2x/4 + C$

1.5 l) CORRECT ANSWER - C

This problem falls under the category of 'Integral Calculus'.

$$\int (\sin^2 x + \cos^2 x)\,dx$$

According to the trigonometric identity given in NCEES® FE Reference Handbook:

$$\sin^2 x + \cos^2 x = 1$$

Substituting this formula in given indefinite integral results in: $\int (\sin^2 x + \cos^2 x)\,dx = \int 1\,dx = x + C$

Thank you

1.9

Discrete Mathematics

Discrete Mathematics

❖ Union

Let A and B be sets. The union of the sets A and B , denoted by **A ∪ B**, is the set that contains those elements that are either in A or in B , or in both.

The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$

❖ Intersection

Let A and B be sets. The intersection of the sets A and B , denoted by **A ∩ B**, is the set that contains those elements that are in both A and B.

The intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 3\}$

❖ Difference

Let A and B be sets. The difference of A and B , denoted by **A − B** , **is the set containing those elements that are in A but not in B.**

$$A = \{1,3,5\}, \qquad B = \{1,2,3\}$$
$$A - B = \{5\}$$

❖ Sets.

A set is **an unordered collection** of objects. The objects in a set are called the **elements,** or **members,** of the set. A set is said to contain its elements.

❖ Disjoint

Two sets are called disjoint if their **intersection** is **the empty set**.

$A \cap B = \emptyset$

❖ Cardinality

The cardinality is the number of distinct elements in S. The cardinality of S is denoted by **|S|** .

$$S = \{a, b, c, d, \{2\}\}$$
$$|S| = 5$$

$$A = \{1, 2, 3, \{2,3\}, 9\}$$
$$|A| = 5$$

$$\{\emptyset\} = \{\{\ \}\}$$
$$|\{\emptyset\}| = 1$$

❖ Empty Set

There is a special set that has no elements. This set is called the **empty set**, or **null set**, and is **denoted by ∅**. **The empty set can also be denoted by { }**

S = {a, b, c, d} We write **a ∈ S** to denote that a is an element of the set S. The notation **e ∉ S** denotes that e is not an element of the set S.

❖ Subset

The set A is said to be a subset of B **if and only if every element of A is also an element of B** . We use the notation **A ⊆ B** to indicate that A is a subset of the set B .

Problem 1.3 b) ______ is a subset of set {a, b, c, d, e}.

(A) {x, y, z}

(B) {a, b, c, d, e}

(C) {a, f, h, e}

(D) None of the above

1.3 b) CORRECT ANSWER - B

This problem falls under the category of 'Discrete Math – Set Theory'.

'Set A' is defined as a subset of 'Set B' if every element in 'Set A' is also present in 'Set B'.

Equal sets are also subsets. According to the definition {a, b, c, d, e} qualifies as a subset of {a, b, c, d, e}.

❖ Proper Subset

The set A is a subset of the set B but that **A ≠ B,** we write **A ⊂ B** and say that A is a proper subset of B.

Problem 1.3 a) ______ is a proper subset of set {2, 4, 6, 8, 10, 12}.

(A) {2, 3, 4, 5}

(B) {2, 4, 6, 8, 10, 12, 14}

(C) {2, 4, 6}

(D) {2, 4, 6, 8, 10, 12}

1.3 a) CORRECT ANSWER - C

This problem falls under the category of 'Discrete Math – Set Theory'.

'Set A' is defined as the proper subset of 'Set B' if every element in 'Set A' is also present in 'Set B', and there exists at least one element in 'Set B' which is not present in 'Set A'.

According to above given definition only {2, 4, 6} qualifies as a proper subset of {2, 4, 6, 8, 10, 12}.

TABLE Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

TABLE Set Identities.

$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Order not important

$$S = \{a,b,c,d\} = \{b,c,a,d\}$$

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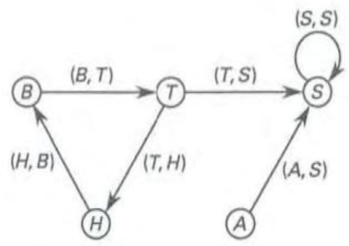
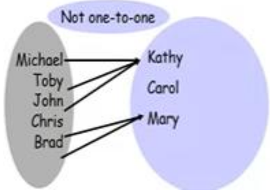
Discrete Mathematics

❖ **GRAPHS OF A RELATION**

A relation between sets can be illustrated by a directed graph (digraph) that connects the set members by arrows, as in Fig. 13.2.

Figure 13.2 Directed Graph

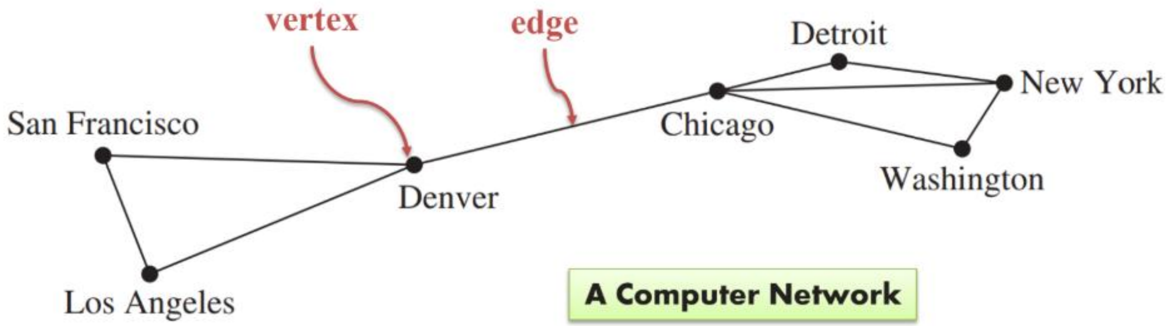




❖ Definition: A graph G = (V, E) consists of V, a nonempty set of vertices (or nodes) and E, a set of edges.

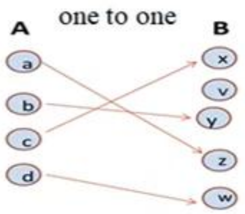
❖ Each edge has either one or two vertices associated with it, called its endpoints.

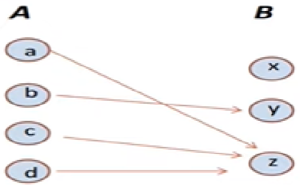
❖ An edge is said to connect its endpoints.

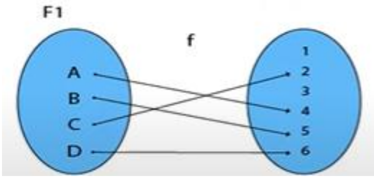


Set functions

❖ An **injective function** results in a unique result for each input. For Finite sets X and Y with discrete elements, injective mapping means that each member of X has a unique, corresponding member of Y. Correspondingly, there is an inverse injective function that maps those corresponding members of Y onto unique members of x.







Problem 1.3 e) Which one of the following relation(s) is not an example of a function?

(A) {(1,b),(1,c),(1,d) }

(B) {(1,a),(2,a),(3,a) }

(C) {(1,a),(2,b),(3,c) }

(D) None of the above

1.3 e) CORRECT ANSWER - A

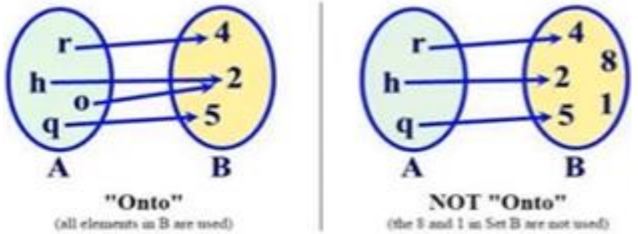
This problem falls under the category of 'Discrete Math – Function Characteristics'.

Function is defined as a set of relations between inputs (domain – x) and outputs (range – y) such that each input is related to only one output.

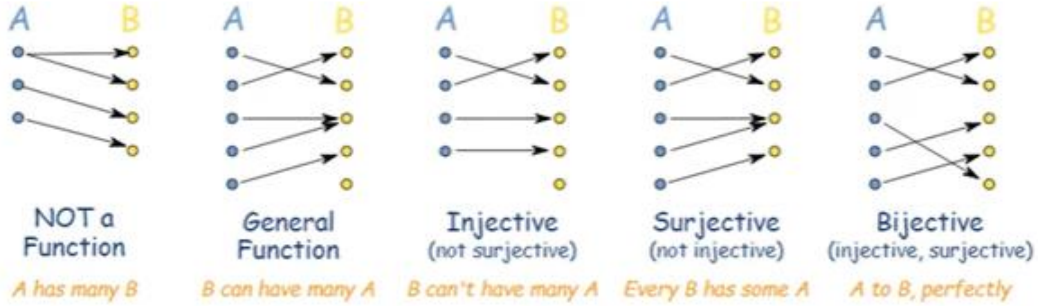
According to the definition, {(1, a), (1, b), (1, c)} is not a function because input 1 is related to multiple outputs.

❖ **SURJECTIVE (ONTO) FUNCTION**

For finite sets X and Y with discrete elements, **a function is surjective if every member of Y can be derived from at least one member of X.** A surjective mapping (function) does not omit any members of Y.



❖ A **bijective function is both injective and surjective**. Each member of X corresponds to a unique member of Y; and, each member of Y corresponds to a unique member of X. A one-to-one relationship exists in both the X-to- Yand Y-to-X mapping directions.



3. Consider two sets, A and B, where Set A has four elements and Set B has five elements. A function $f(x)$ that maps Set A to Set B, where each element of A is mapped to a unique element of B, is:

Select **all** that apply:

- ☐ A. injective
- ☐ B. surjective
- ☐ C. bijective
- ☐ D. the inverse of the function mapping B to A
- ☐ E. an invalid general function

3. The definitions of injective, surjective, and bijective functions are given in the Discrete Math section in the Mathematics chapter of the *FE Reference Handbook*.

Since no element of B is a function of more than a single element of A, there is a one-to-one (i.e., injective) relationship from A to B. $f(x)$ cannot be surjective since at least one element of B does not map from any element of A. Since it cannot be surjective, it is, by definition, not bijective. Since there is a one-to-one mapping from A to B, the inverse of $f(x)$, or $f^{-1}(x)$, maps B to A. The function is a valid general function since no element of A maps to more than one element of B.

THE CORRECT ANSWERS ARE: A, D

Problem 1.3 f) {(a,1),(b,1),(c,2),(d,2)} is an example of _______ function.

(A) Injective

(B) Surjective

(C) Bijective

(D) It's not a function

1.3 f) CORRECT ANSWER - B

This problem falls under the category of 'Discrete Math – Function Characteristics'.

{(a, 1), (b, 1), (c, 2), (d, 2)} is a surjective function because each output is linked to at least one input.

Helpful tip – Understand the difference between injective, surjective and bijective functions.

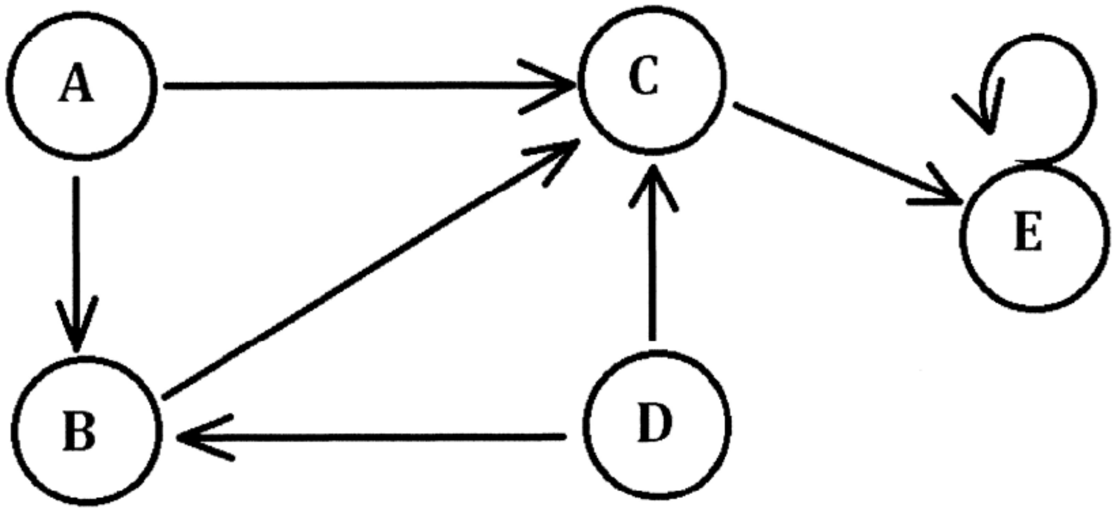
Problem 1.3 l) __________ correctly represents the set of vertices for following directed graph?

(A) {A, B, C, D}

(B) {A, B, C, D, E}

(C) {A, B, C, D, E, E}

(D) None of the above



1.3 l) CORRECT ANSWER – B

It can be observed that given directed graph has 5 nodes/vertices in the form of A, B, C, D and E.

Problem 1.3 m) __________ correctly represents the set of edges for above given directed graph?

(A) {(A,C),(A,B),(B,C),(D,B),(D,C),(C,E),(E,E)}

(B) {(A,C),(B,A),(C,B),(D,B),(D,C),(C,E),(E,E)}

(C) {(A,C),(A,B),(B,C),(B,D),(C,D),(C,E),(E,E)}

(D) {(A,C),(A,B),(B,C),(D,B),(C,D),(E,C),(E,E)}

1.3 m) CORRECT ANSWER – A

The given directed graph contains following edges/arcs:

$A \rightarrow C\ (A, C)$

$A \rightarrow B\ (A, B)$

$B \rightarrow C\ (B, C)$

$D \rightarrow B\ (D, B)$

$D \rightarrow C\ (D, C)$

$C \rightarrow E\ (C, E)$

$E \rightarrow E\ (E, E)$

Thank you